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Characterizing the dynamics of three FitzHugh-Nagumo neurons network

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Abstract. The behavior of neuron systems can be modeled by the FitzHugh-Nagumo model, originally consisting of two nonlinear differential equations, which simulates the behavior of nerve impulse conduction through the neuronal membrane. In this work, we numerically study the dynamical behavior of three coupled neurons in a network modeled by the FitzHugh-Nagumo equations. We consider three neurons coupled unidirectionally and bidirectionally, for which Lyapunov diagrams were constructed calculating the Lyapunov exponents. The coupling parameter between neurons has an important role to the understanding of the physiological mechanisms of the nervous system. In this sense, the dynamics of the neural networks here investigated are presented in terms of the variation between the coupling strength of the neurons and other parameters of the system. The results show the occurrence of periodic structures embedded in chaotic regions, and also the existence of hyperchaos in their dynamics, besides, we show the importance of the type of coupling between the neurons, with respect to the existence of those behaviors for the same parameter set.

Keywords. FitzHugh-Nagumo networks, Lyapunov diagrams, Periodic structure, Lyapunov exponent, Hyperchaos

1 Introduction

The nervous system is composed by cells denominated neurons responsible for receive and interpret information that are used to maintain the body functions. The operation of the whole nervous system is based on the chemical and electrical synapses [12]. Nowadays, the efforts of researchers go beyond to the simple understanding of the neural systems functions, and researches to discover efficient treatments of neurodegenerative diseases are being carried out [11]. In the recent paper [11], the authors show that the possibility to decoupling neurons can be an efficient strategy to neuroprotection and prevent the process of cell deaths related to the neurodegenerative diseases as Parkinson, Alzheimer, epilepsy, for instance. In this context, the coupling of neurons in a network is an important

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parameter that must be taken into account, if we want to model neural networks. In the literature there are several models to describe and simulate networks of neurons [5, 8, 10].

Under this scenario, the Chaos Theory becomes present by its relevance and some success in explain many features shown by neural systems [6, 8, 15]. More recently, a growing interest is in the study of the dynamics on the parameter-spaces of the neural system models [7, 16]. The parameter-spaces are colorful diagrams when two parameters of the model are varied and the other are kept constant. The colors codify some measure that can be computed on the model. Usually, this measure is the Lyapunov exponent [1, 3, 7, 9, 13, 14], periods [7, 14], or other invariant measure [3]. This procedure allows us to identify regions of periodic, chaotic, and hyperchaotic behavior, and recently it is applied in several models. A general feature is observed in these parameter-spaces, the existence of *shrimp*-shaped periodic structures embedded in chaotic domains [7, 14]. These structures are stable periodic domains, *i.e.*, inside them the system variables oscillate periodically with a well-defined period, and often bordered by chaotic regions.

Our main goal in this paper is to investigate the dynamics of three coupled FitzHugh-Nagumo (FHN) neurons network regarding the type of coupling. The dynamics is investigated through the parameter-spaces and we compare their structural changes according to the type of coupling. As mentioned above, due to the relevance of the coupling parameter between the neurons of a network, all the parameter-spaces presented here are in relation to this parameter. The role of the coupling in FHN-networks was studied in recent papers [2, 4], but with different approaches of those studied here. In our work we carry out a systematic study of the FHN-network with respect to the coupling type.

2 FitzHugh-Nagumo neural network models

In this section we introduce the FHN-network models studied in this paper for the two types of coupling, namely unidirectional and bidirectional coupling, with $n = 3$ neurons. The case for $n = 2$ was recently published elsewhere [7]. In the bidirectional coupling the neurons interact mutually, all to all system, while for unidirectional coupling the neuron i interacts only with the neuron $i + 1$.

For the unidirectional coupling with $n > 2$ coupled neural units, or neurons, we have the following generalized FHN-network model

$$\begin{aligned}\frac{dx_i}{dt} &= c(y_i + x_i - \frac{1}{3}x_i^3) + \gamma(x_i - x_{i+1}), \qquad (1)\\ \frac{dy_i}{dt} &= -1/c(x_i - a + by_i),\end{aligned}$$

with the following boundary condition: $x_{n+1} = x_1$. x_i and y_i, with $i = 1, ..., n$, represent the qualitative behaviors of the voltage variable across the cell membrane, and of the recovery variable of the resting membrane, respectively. a, b, and c are tunable parameters of the model, and γ is the coupling strength between the neurons.

For bidirectional coupling with n coupled neural units, or neurons, the generalized FHN-network model can be written as

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$$\begin{aligned}\frac{dx_i}{dt} &= c(y_i + x_i - \frac{1}{3}x_i^3) + \gamma\sum_{j=1}^{n}(x_i - x_j), \qquad (2)\\ \frac{dy_i}{dt} &= -1/c(x_i - a + by_i),\end{aligned}$$

for i, and $j = 1, ..., n$. The variables x_i and y_i, and parameters a, b, c, and γ have the same meanings of those shown in system 1.

In this work we study two FHN-network models for $n = 3$, one obtained by system 1, and other obtained by system 2. In the next section we present the results obtained from the numerical solutions of these equations whose behaviors depend on four parameters, namely a, b, c, and γ. The results are presented by colorful two-dimensional diagrams, and we use the Lyapunov exponents as the invariant measure to plot the parameter-spaces, here namely as Lyapunov diagrams.

3 Lyapunov diagrams

The Lyapunov diagram plots are obtained for the following parameter combinations: $a \times \gamma$, $b \times \gamma$, and $c \times \gamma$, for the systems 1, and 2. The Lyapunov diagrams were constructed using the Lyapunov exponents (LEs) spectrum numerically calculated for these systems. To obtain the LEs spectrum, the systems are numerically solved by the Runge-Kutta method with time step 1×10^{-1} and 5×10^5 iterations to obtain the LEs spectrum for each parameters pair discretized in a grid of 500×500 values. Therefore, we obtain 2.5×10^5 LEs spectra for each parameter pair, where each spectrum has the number of values of LEs equal to the number of dimensions of the system. For instance, for any N-dimensional coupled first-order differential equations, which one has N values of LEs in the spectrum, for each set of fixed parameters.

For FHN-network with three neurons unidirectionally coupled, system 1 with $n = 3$, we show in Fig. 1 the Lyapunov diagrams for the following combinations of parameters: $\gamma \times a$, Fig. 1(a); $\gamma \times b$, Fig. 1(b); and $\gamma \times c$, Fig. 1(c). In this case, the LEs spectrum include six values. The diagrams in Fig. 1, were plotted with the largest exponent of the LEs spectrum. The initial conditions are $(x_1, y_1, x_2, y_2, x_3, y_3) = (-0.1, 0.5, 0.1, -0.3, 0.1, 0.3)$. All the diagrams in Fig. 1 present periodic structures embedded in chaotic regions, as also observed for the two-neurons FHN-networks. Both cases, two- and three-neurons, retain this feature in the Lyapunov diagrams. In the diagrams shown in Fig. 1(a) and (b), we observe that they are similar and for γ, coupling parameter, between 0.012 and 0.17 the chaotic region is confined by a large periodic region. In the projection of Fig. 1(c), it is not clear the presence of periodic structures.

Another feature of the system 1 with $n = 3$, is the presence of hyperchaotic behavior, feature not found in the two-neuron FHN-network models (see Ref. [7]). To observe hyperchaotic behavior in the Lyapunov diagrams, we plot the parameter-space with the second largest exponent of the LEs spectrum. If there are positive second largest exponents, the parameter-space will show yellow-red-blue colors. Indeed, in Figs. 2(a) and (b) we show

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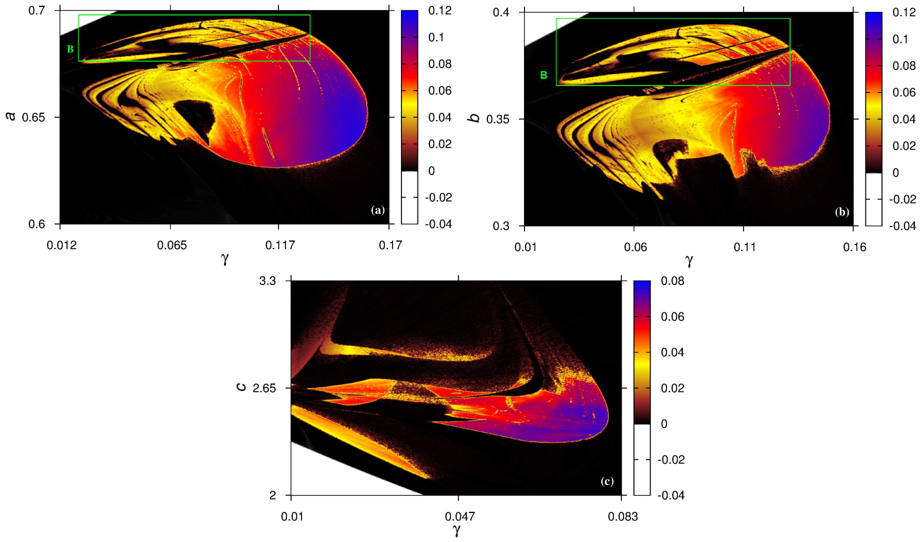

Figure 1: Lyapunov diagrams for the largest exponents of the LEs spectra codified by colors, as the color bar at the right side, for three neurons with unidirectional coupling. (a) $\gamma \times a$ plane, with $b = 0.4$ and $c = 2.0$. (b) $\gamma \times b$ plane with $a = 0.7$ and $c = 2.0$. (c) $\gamma \times c$ plane with $a = 0.7$ and $b = 0.4$. Periodic behaviors are represented by black color, chaotic ones are yellow-red-blue colors, and white color is fixed point region.

the Lyapunov diagrams for the second largest exponents, for the regions delimited by the green boxes B in Figs. 1(a) and (b), respectively. In these diagrams, red color suggests hyperchaotic behavior of system 1 with $n = 3$. Qualitatively, this suggests that the dynamical variables x_i and y_i of the three-neurons FHN-network with unidirectional coupling present hyperchaotic oscillations, behavior more complex than chaotic oscillations, by the presence of two positive Lyapunov exponent.

In Fig. 3 we present the Lyapunov diagrams for the largest exponents of the spectra for three neurons ($n = 3$) with bidirectional coupling, model described by system 2. As the unidirectional case, the spectrum of the LEs of the system 2 has six exponents, and the diagrams shown in Fig. 3 were constructed with the largest one. Again, periodic structures embedded in chaotic region also are observed. In both cases, two- and three-neurons, retain this feature in the Lyapunov diagrams. In the diagrams shown in Figs. 3(a) and (b), we observe that they are similar and for γ, coupling parameter, greater than 0.122, there are no more periodic structures and an open chaotic region (bluish region) emerged, the opposite was observed in Fig. 1, where the chaotic region is confined by periodic regions. This is the primary difference between three-neurons FHN-networks with uni and bidirectional coupling. A qualitative interpretation regarding the dynamical variables x_i and y_i is that, for unidirectional coupling, these variables oscillates chaotically enclosed in the range of 0.012 and 0.17 for the coupling parameter, γ, and for bidirectional case, these variables oscillates chaotically unbounded for $\gamma > 0.122$, because there are no periodic structures above this value.

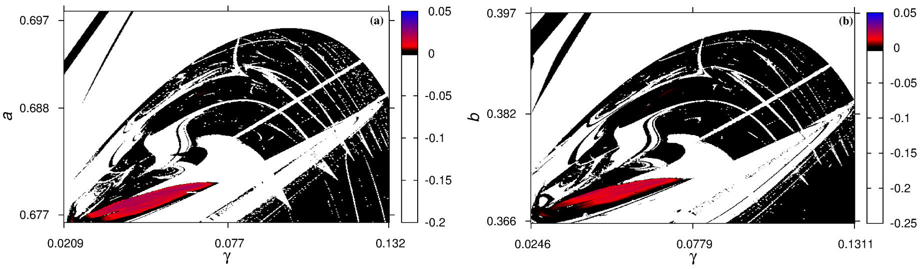

Figure 2: (a) Lyapunov diagram of the second largest exponent, for the region of the green box B in Fig. 1(a). (b) Lyapunov diagram of the second largest exponent, for the region of the green box B in Fig. 1(b). Here white color suggests fixed point or periodic behavior, black color suggests quasi-periodic (torus) or chaotic behavior, and red color implies hyperchaotic behavior.

4 Conclusions

In this paper we have investigated the dynamics of neural networks composed by three neurons, using unidirectional and bidirectional coupling. Each neuron is modeled by a set of two autonomous nonlinear first-order ordinary differential equations that describes the propagations of a nerve impulse through the neuronal membrane, namely FitzHugh-Nagumo (FHN) model. Our main goal was the investigation of the influence of the coupling strength between three neurons with the type of coupling. For this purpose, we constructed the Lyapunov diagrams, which are parameter-spaces with the Lyapunov exponents codified by colors, for the two cases of the FHN-network. By contrasting the Lyapunov diagrams in each case, we observe the changes in the dynamical behaviors of these networks.

For networks of three neurons, and for both coupling cases, the Lyapunov diagrams are rather different over wide regions (see Figs. 1, and 3) but a general feature was observed in some regions of these diagrams, namely the existence of periodic structures embedded in chaotic regions. These sets of periodic structures are presented in a wide range of nonlinear systems [14]. An exception is in high-dimensional systems with more than three-dimensions, where hyperchaotic behaviors can occur. In these systems, on the hyperchaotic regions there are no periodic structures or the structures are malformed or shapeless [17]. This feature occurs for three-neurons FHN-network with unidirectional coupling (see Figs. 1 and 2). Therefore, we remark that from $n = 3$ the Lyapunov diagrams show a more complex structure than $n = 2$ published in the Ref. [7], due to the emergence of hyperchaos. Moreover, from $n = 2$ to $n = 3$ the Lyapunov diagrams show that the dynamics becomes increasingly more complex. We believe that this apparent pattern remains for higher n until the Lyapunov diagrams just present chaotic domains. Further investigations could be interesting in this direction.

The coupling strength parameter (γ) represents the connections between the network neurons, and it has an important role for the flow of ions through the gap junctions between these neurons. In this sense, we showed that γ also has an important role in the dynamics of the FHN-networks. Depending on its value, and on the combination of values of the

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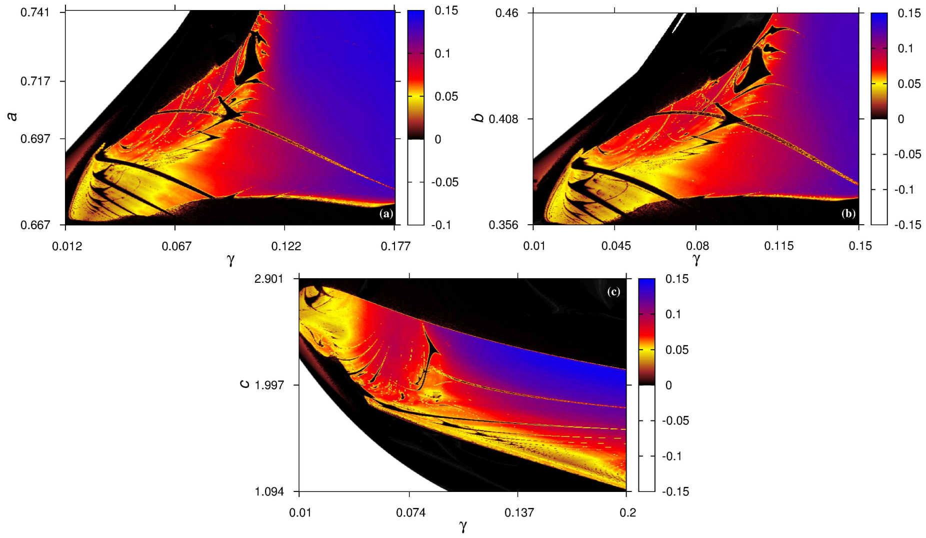

Figure 3: Lyapunov diagrams for the largest exponents of the LEs spectra codified by colors, as the color bar at the right side, for three neurons with bidirectional coupling. (a) $\gamma \times a$ plane, with $b = 0.4$ and $c = 2.0$. (b) $\gamma \times b$ plane with $a = 0.7$ and $c = 2.0$. (c) $\gamma \times c$ plane with $a = 0.7$ and $b = 0.4$. Periodic behaviors are represented by black color, chaotic ones are yellow-red-blue colors, and white color is fixed point region.

adjustable parameters, namely a, b, and c, besides the type of coupling, the FHN-network variables x_i and y_i can show periodic, chaotic and hyperchaotic oscillations, which means that the neural network as a whole can presents a steady state or a complex interactions with the environment.

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