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Analytical solutions for optimal low-thrust limited-power transfers between coplanar orbits

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Abstract. In this paper, analytical solutions, which include the short periodic terms, for the problem of optimal time-fixed low-thrust limited power transfers (no rendezvous), in an inverse-square force field, between coplanar orbits are revisited. These solutions are expressed in classical orbital elements for transfers between elliptical orbits and in nonsingular elements for transfers between orbits with small eccentricities. In both cases, the solutions are derived through canonical transformation theory. A brief discussion about the solution of the two-point boundary value problem of going from an initial orbit to a final orbit at the prescribed final time, based on the analytical solutions, is also presented.

Keywords. Optimal low-thrust transfers, optimal space trajectories, transfers between coplanar orbits.

1 Introduction

The main purpose of this work is to revisit analytical solutions for optimal low-thrust limited-power transfers between coplanar orbits in an inverse-square force field. This study has been motivated by the renewed interest in the use of low-thrust propulsion systems in space missions in the last thirty years. Important space missions such as NASA-JPL Deep Space 1 and ESA-SMART1 have made use of low-thrust propulsion systems. Low-thrust electric propulsion systems are characterized by high specific impulse and low-thrust capability (the ratio between the maximum thrust acceleration and the gravity acceleration on the ground is small, between 10^{-4} and 10^{-2}) and have their greatest benefits for high-energy planetary missions. Several researchers have obtained numerical and analytical solutions for several maneuvers involving specific initial and final orbits and specific thrust profiles. In the analytical studies,

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averaging techniques and perturbation methods are applied and analytical solutions of the averaged equations, as well as first order solutions which include short periodic terms, are obtained [1-5].

The authors have presented a complete analytical solution, which includes the short periodic terms, for the problem of optimal low-thrust limited-power transfers between arbitrary elliptical coplanar orbits [2] and for transfers between orbits with small eccentricities [1] by using a canonical approach based on canonical transformation theory, including the Hamilton-Jacobi theory and a perturbation method based on Lie series. Classical orbital elements are used for describing the analytical solution in the case of transfers between elliptical orbits, and, a suitable set of nonsingular elements are used in the case of transfers between orbits with small eccentricities. These first order analytical solutions for time-fixed transfers between coplanar orbits are then revisited and numerical results are presented for two arbitrary missions.

2 Formulation of the optimization problem

For a low-thrust limited-power propulsion system – LP system, the fuel consumption is described by the variable J defined as [5]

$$J = \frac{1}{2}\int_{t_0}^{t} \gamma^2 dt\,, \tag{1}$$

where γ is the magnitude of the thrust acceleration vector $\boldsymbol{\gamma}$, used as control variable. The consumption variable J is a monotonic decreasing function of the mass m of the space vehicle, such that the minimization of J_f is equivalent to the maximization of m_f.

Consider the motion of a space vehicle **M**, powered by a limited-power engine in an inverse-square force field. At time t, the state of a space vehicle **M** is defined by the radial distance r from the center of attraction, the radial and circumferential components of the velocity, v_r and v_s, and the fuel consumption J.

The optimization problem can be formulated as a Mayer problem of optimal control as follows [5]: It is proposed to transfer the space vehicle **M** from the initial state $(r_0, v_{r0}, v_{s0}, 0)$ at time t_0 to the final state $(r_f, v_{rf}, v_{sf}, J_f)$ at time t_f, such that the final consumption variable J_f is a minimum. The duration of the transfer $t_f - t_0$ is specified. In the two-dimensional formulation, the state equations are given by

$$\frac{dr}{dt} = v_r \qquad \frac{dv_r}{dt} = \frac{v_s^2}{r} - \frac{\mu}{r^2} + R \qquad \frac{dv_s}{dt} = -\frac{v_r v_s}{r} + S \qquad \frac{dJ}{dt} = \frac{1}{2}\left(R^2 + S^2\right), \tag{2}$$

where μ is the gravitational parameter, R and S are the radial and circumferential components of the thrust acceleration vector, respectively. The performance index is

$$IP = J(t_f)\,. \tag{3}$$

For LP system, there are no constraints on the thrust acceleration vector.

According to the previous work [1], the maximum Hamiltonian H^* is defined by

$$H^* = v_r p_r + \left(\frac{v_s^2}{r} - \frac{\mu}{r^2}\right) p_{v_r} - \frac{v_r v_s}{r} p_{v_s} + \frac{1}{2}\left(p_{v_r}^2 + p_{v_s}^2\right), \tag{4}$$

and the optimal thrust acceleration is given by

$$R^* = p_{v_r} \qquad S^* = p_{v_s} \,. \tag{5}$$

3 Analytical Solutions

A first order analytical solution for the system of differential equations governed by the maximum Hamiltonian, which includes short periodic terms, can be derived applying canonical transformation theory, as described in [2] for transfers between elliptical orbits and in [1] for transfers between orbits with small eccentricities.

The optimal low-thrust power-limited trajectories for transfers between coplanar elliptical orbits are then described in a set of classical orbital elements by the following equations:

$$a(t) = a'(t) + \sqrt{\frac{a'^5}{\mu^3}} \left[8e' \sin E' a'^2 p'_a + 4\left(1-e'^2\right) \sin E' a' p'_e - 4\frac{\left(1-e'^2\right)^{1/2}}{e'} \cos E' a' p'_\omega \right]_{E'_0}^{E'} , \tag{6}$$

$$\begin{aligned} e(t) = e'(t) + \sqrt{\frac{a'^5}{\mu^3}} \Bigg[& 4\left(1-e'^2\right) \sin E' a' p'_a + \left(1-e'^2\right)\left[-\frac{5}{4} e' \sin E' + \frac{3}{4}\sin 2E' - \frac{1}{12} e' \sin 3E' \right] p'_e \\ & + \frac{\left(1-e'^2\right)^{1/2}}{e'} \left[\frac{5}{4} e' \cos E' + \frac{1}{4}\left(e'^2 - 3\right) \cos 2E' + \frac{1}{12} e' \cos 3E' \right] p'_\omega \Bigg]_{E'_0}^{E'} , \end{aligned} \tag{7}$$

$$\begin{aligned} \omega(t) = \omega'(t) + \sqrt{\frac{a'^5}{\mu^3}} \Bigg[& -4\frac{\left(1-e'^2\right)^{1/2}}{e'} \cos E' a' p'_a + \frac{\left(1-e'^2\right)^{1/2}}{e'} \left[\frac{5}{4} e' \cos E' + \frac{1}{4}\left(e'^2 - 3\right)\cos 2E' \right. \\ & \left. + \frac{1}{12} e' \cos 3E' \right] p'_e + \frac{1}{e'^2} \left[\left(\frac{5}{4} - e'^2 \right) e' \sin E' - \frac{1}{2}\left(\frac{3}{2} - e'^2 \right) \sin 2E' + \frac{1}{12} e' \sin 3E' \right] p'_\omega \Bigg]_{E'_0}^{E'} , \end{aligned} \tag{8}$$

with a', e', …, p'_ω given through the following equations

$$a' = a'' \qquad e' = \sin\varphi \qquad \omega' = \omega'' \qquad p'_a = p''_a \qquad p'_e = \frac{p_\varphi}{\cos\varphi} \qquad p'_\omega = p''_\omega ,$$

and,

$$a''(t) = \frac{a''_0}{1 + \dfrac{4a''_0}{\mu}\left(\dfrac{1}{2}\mathrm{E}t^2 - a''_0 p''_{a_0} t \right)} \qquad p''^2_a = \left(\frac{a''_0}{a''} \right)^3 p''^2_{a_0} + \frac{1}{8} p''^2_{\omega_0} \left(5\csc^2 k_1 - 4 \right)\left(\frac{a''_0}{a''^3} - \frac{1}{a''^2} \right)$$

$$a'' \sin^2 k_0 = a''_0 \sin^2\left(\sqrt{2}\psi + k_0 \right) \qquad \cos\varphi = \cos k_1 \cos\tau$$

$$\psi = \frac{1}{5}\left(\tau - \tau_0 \right)\sqrt{1 + 4\cos^2 k_1} \qquad \omega'' = k_2 + \tan^{-1}\left(\tan\tau \csc k_1 \right) - \frac{4}{5}\tau \sin k_1$$

$$p^2_\varphi = p''^2_{\omega_0}\left(\csc^2 k_1 - \csc^2\varphi \right) \qquad p''_\omega = p''_{\omega_0} ,$$

with the auxiliary constants k_0, k_1 and k_2 defined as functions of the initial value of the adjoint variables by

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$$\csc^2 k_0 = \frac{8\left(a''_0 p''_{a_0}\right)^2 + p''^2_{\omega_0}\left(5\csc^2 k_1 - 4\right)}{p''^2_{\omega_0}\left(5\csc^2 k_1 - 4\right)} \qquad \csc^2 k_1 = \frac{p^2_{\varphi_0} + p''^2_{\omega_0}\csc^2\varphi_0}{p''^2_{\omega_0}}$$

$$k_2 = \omega''_0 + \frac{4}{5}\tau_0 \sin k_1 - \tan^{-1}\left(\tan\tau_0 \csc k_1\right).$$

The constants C, C_1, C_2 and E can also be written as functions of the initial value of the adjoint variables:

$$C^2 = \frac{1}{5} p''^2_{\omega_0}\left(5\csc^2 k_1 - 4\right) \qquad C_1 = p''_{\omega_0}$$

$$C_2^2 = p^2_{\varphi_0} + p''^2_{\omega_0}\csc^2\varphi_0 \qquad 4\mu \mathrm{E} = a''_0\left(8\left(a''_0 p''_{a_0}\right)^2 + p''^2_{\omega_0}\left(5\csc^2 k_1 - 4\right)\right).$$

The initial conditions are defined by $a''(0) = a''_0$, $e''(0) = \sin\varphi_0$ and $\omega''(0) = \omega''_0$, and, τ_0 is obtained from $\cos\varphi_0 = \cos k_1 \cos\tau_0$. The eccentric anomaly E' is computed from Kepler´s equation with the mean anomaly M' given by

$$M'(t) = M'(t_0) + \int_{t_0}^{t}\left[\sqrt{\frac{\mu}{a'^3}} - \left(\frac{5+2e'^2}{2}\right)\sqrt{\frac{a'^5}{\mu^3}}\frac{\sqrt{1-e'^2}}{e'^2} p'_\omega\right]dt\ .$$

The optimal low-thrust power-limited trajectories for transfers between coplanar orbits with small eccentricities are then described by the following equations:

$$\begin{aligned} a''(t) = a'''(t) + \sqrt{\frac{a'''^5}{\mu^3}}\Big\{&\left[8h'''\sin\ell''' - 8k'''\cos\ell'''\right]a'''^2 p_{a'''} + \left[4a'''\sin\ell''' - 2a'''k'''\cos 2\ell'''\right.\\ &\left.+2a'''h'''\sin 2\ell'''\right]p_{h'''} + \left[-4a'''\cos\ell''' - 2a'''h'''\cos 2\ell''' - 2a'''k'''\sin 2\ell'''\right]p_{k'''}\Big\}\Big|_{t_0}^{t} \end{aligned} \tag{9}$$

$$\begin{aligned} h''(t) = h'''(t) + \sqrt{\frac{a'''^5}{\mu^3}}\Big\{&\left[4a'''\sin\ell''' - 2a'''k'''\cos 2\ell''' + 2a'''h'''\sin 2\ell'''\right]p_{a'''}\\ &+\left[-2h'''\sin\ell''' - 2k'''\cos\ell''' + \frac{3}{4}\sin 2\ell''' + \frac{2}{3}h'''\sin 3\ell''' - \frac{2}{3}k'''\cos 3\ell'''\right]p_{h'''}\\ &+\left[2h'''\cos\ell''' - 2k'''\sin\ell''' - \frac{3}{4}\cos 2\ell''' - \frac{2}{3}h'''\cos 3\ell''' - \frac{2}{3}k'''\sin 3\ell'''\right]p_{k'''}\Big\}\Big|_{t_0}^{t} \end{aligned} \tag{10}$$

$$\begin{aligned} k''(t) = k'''(t) + \sqrt{\frac{a'''^5}{\mu^3}}\Big\{&\left[-4a'''\cos\ell''' - 2a'''h'''\cos 2\ell''' - 2a'''k'''\sin 2\ell'''\right]p_{a'''}\\ &+\left[2h'''\cos\ell''' - 2k'''\sin\ell''' - \frac{3}{4}\cos 2\ell''' - \frac{2}{3}h'''\cos 3\ell''' - \frac{2}{3}k'''\sin 3\ell'''\right]p_{h'''}\\ &+\left[2h'''\sin\ell''' + 2k'''\cos\ell''' - \frac{3}{4}\sin 2\ell''' - \frac{2}{3}h'''\sin 3\ell''' + \frac{2}{3}k'''\cos 3\ell'''\right]p_{k'''}\Big\}\Big|_{t_0}^{t} \end{aligned} \tag{11}$$

with a''', …, $p_{k'''}$ given by

$$a'''(t) = \frac{a'''_0}{1 + \dfrac{4a'''_0}{\mu}\left(\dfrac{1}{2}\mathrm{E}t^2 - a'''_0 p_{a'''_0} t\right)} \qquad p^2_{a'''} = \left(\frac{a'''_0}{a'''}\right)^3 p^2_{a'''_0} + \frac{5}{8}C^2\left(\frac{a'''_0}{a'''^3} - \frac{1}{a'''^2}\right)$$

$$h'''(t) = h_0''' + \sqrt{\frac{5}{2}}\frac{p_{h'''}}{C}\left\{\tan^{-1}\left(\left(\frac{4\mu \mathrm{E}}{5C^2 a_0'''}-1\right)^{1/2}\right) - \tan^{-1}\left(\left(\frac{4\mu \mathrm{E}}{5C^2 a'''}-1\right)^{1/2}\right)\right\} \qquad p_{h'''} = p_{h_0'''}$$

$$k'''(t) = k_0''' + \sqrt{\frac{5}{2}}\frac{p_{k'''}}{C}\left\{\tan^{-1}\left(\left(\frac{4\mu \mathrm{E}}{5C^2 a_0'''}-1\right)^{1/2}\right) - \tan^{-1}\left(\left(\frac{4\mu \mathrm{E}}{5C^2 a'''}-1\right)^{1/2}\right)\right\} \qquad p_{k'''} = p_{k_0'''},$$

with C and E now given in terms of the initial conditions by

$$C^2 = p_{h_0'''}^2 + p_{k_0'''}^2 \qquad 4\mu \mathrm{E} = a_0'''\left(8\left(a_0''' p_{a_0'''}\right)^2 + 5\left(p_{h_0'''}^2 + p_{k_0'''}^2\right)\right)$$

The initial conditions for the state variables are $a'''(0) = a_0'''$, $h'''(0) = h_0'''$ and $k'''(0) = k_0'''$. Note that the mean latitude ℓ''' in equations above is given by

$$\ell'''(t) = \ell'''(t_0) + \int_{t_0}^{t}\left[\sqrt{\frac{\mu}{a'''^3}} + \frac{7}{4}\frac{a'''}{\mu}\left(k''' p_{h'''} - h''' p_{k'''}\right)\right]dt.$$

The set of nonsingular orbital elements are related to the classical orbital elements through the equations

$$h = e\cos\omega \qquad k = e\sin\omega \qquad \ell = M + \omega,$$

where e is the eccentricity, ω is the argument of periapsis and M is the mean anomaly, and, a denotes the semi-major axis. For simplicity, primes are omitted in the above equations. In both solutions, primes are used to denote new variables introduced through the canonical transformations built to obtain the analytical solution.

The consumption is given by $J = \mathrm{E}(t - t_0) + \Delta S_1$, with $\Delta S_1 = S_1(t) - S_1(t_0)$, where S_1 is the generating function built through Hori method in both cases.

4 Solution of the Two-Point Boundary Value Problem

In this section, an iterative algorithm based on the complete first order analytical solutions, is briefly described for solving the two-point boundary value problem of going from an initial orbit O_0 to a final orbit O_f at the prescribed final time t_f.

For a given final time t_f, the set of Eqns. (6), (7) and (8), can be represented as

$$y_i(t_f) = g_i\left(t, p_{a_0'}, p_{e_0'}, p_{\omega_0'}\right), i = 1,2,3,$$

where $y_1(t_f) = a(t_f)$, $y_2(t_f) = e(t_f)$ and $y_3(t_f) = \omega(t_f)$. Note that $p_{a_0'}$, $p_{e_0'}$ and $p_{\omega_0'}$ appear explicitly in the short periodic terms and also implicitly through $a''(t)$, $e''(t)$, $\omega''(t)$ and $M''(t)$. Thus, the functions g_i, $i = 1,2,3$, are nonlinear in these variables. Similarly, the set of Eqns. (9), (10) and (11), can be represented as

$$y_i(t_f) = g_i\left(t, p_{a_0'''}, p_{h_0'''}, p_{k_0'''}\right),\ i = 4,5,6$$

where $y_4(t_f) = a''(t_f)$, $y_5(t_f) = h''(t_f)$ and $y_6(t_f) = k''(t_f)$. As before, $p_{a_0'''}$, $p_{h_0'''}$ and $p_{k_0'''}$ appear explicitly in the short periodic terms and also implicitly through $a'''(t)$, $h'''(t)$,

$k'''(t)$ and $\ell'''(t)$. Thus, the functions g_i, $i = 4,5,6$, are also nonlinear in these variables.

So, the two-point boundary value problem can be stated as: find the initial adjoint variables such that the prescribed final values of the orbital – classical or nonsingular – elements are satisfied. This boundary value problem can be solved through a Newton-Raphson algorithm.

5 Results

In what follows, the analytical solutions described in the preceding sections are applied in the analysis of two missions considering time-fixed low-thrust transfers between coplanar orbits. The first one of these missions involves the transfer from Earth to Venus for several times of flight, and, the following assumptions are considered: 1. the orbits of the planets are ellipses with small eccentricities; 2. the orbits of the planets lie in the plane of the ecliptic, i.e. the inclination of the orbital plane of Venus is neglected; 3. the flight of the space vehicle takes place in the plane of the ecliptic; 4. only the heliocentric phase is considered; that is, the attraction of planets on the spacecraft is neglected. The orbital elements of Earth and Venus are given in Table 1 (Transfer 1).

The second mission considers an arbitrary transfer with orbital elements of the initial orbit O_0 and of the final orbit O_f also defined in Table 1 (canonical units are used).

The values of the consumption variable *J* are presented in Table 2. In both cases, one sees that the fuel consumption decreases with the flight time. Figure 1 shows the time evolution of semi-major axis, eccentricity and the argument of periapsis for the second mission. Note the contribution of the periodic terms.

Table 1: Orbital elements of the initial and final orbits.

Transfer	Orbit	Orbital Elements		
		Semi-major axis	Eccentricity	Argument of periapsis (º)
1	O_0	1.000	0.0167	102.937
	O_f	0.723	0.0068	131.563
2	O_0	1.000	0.2000	15.00
	O_f	0.750	0.1000	45.00

Table 2: Consumption variable *J*.

$t_f - t_0$	25.0	50.0	75.0	100.0	125.0	150.0
Transfer 1	6.169×10^{-4}	3.088×10^{-4}	2.060×10^{-4}	$1.546\times10\text{-}^{4}$	1.237×10^{-4}	1.031×10^{-4}
Transfer 2	6.341×10^{-4}	3.145×10^{-4}	2.091×10^{-4}	1.566×10^{-4}	1.252×10^{-4}	1.042×10^{-4}

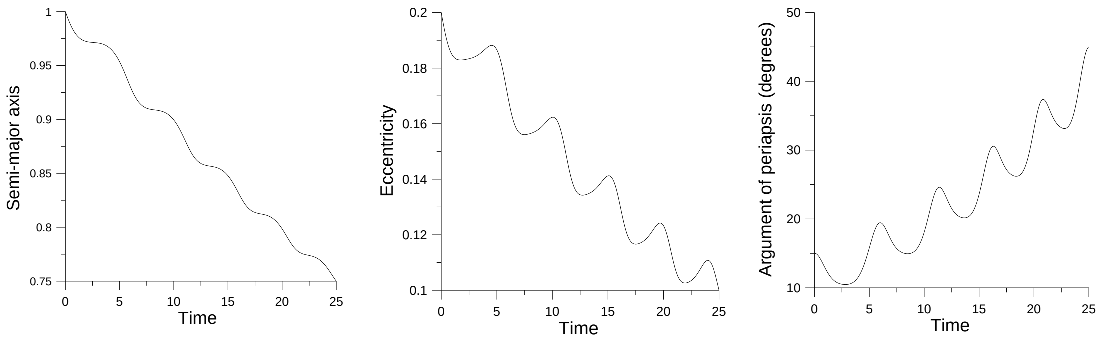

Figure 1 - Time evolution of semi-major axis, eccentricity and argument of periapsis for Transfer 2.

6 Conclusion

In this paper, analytical solutions for optimal time-fixed low-thrust limited power transfers (no rendezvous), in an inverse-square force field, between coplanar orbits are revisited and a brief discussion about the solution of the boundary value problem of going from an initial orbit to a final orbit at the prescribed final time is also presented. Two missions are analyzed and the contribution of the periodic terms is highlighted.

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