Proceeding Series of the Brazilian Society of Applied and Computational Mathematics, Vol. 5, N. 1, 2017.

Trabalho apresentado no CNMAC, Gramado - RS, 2016.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

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Proper and Improper Riemann Integral in a Single Definition

Tiago S. dos Reis[1]
Federal Institute of Education, Science and Technology of Rio de Janeiro, Volta Redonda, RJ
Program of History of Science, Technique, and Epistemology, Federal University of Rio de Janeiro, Rio de Janeiro, RJ

Abstract. This work proposes a new form of integral which arises from infinite partitions. It uses upper and lower series instead of upper and lower Darboux finite sums. It is shown that every Riemann integrable function, both proper and improper, is integrable in the sense proposed here and both integrals have the same value. Furthermore it is shown that the Riemann integral and our integral are equivalent for bounded functions in bounded intervals. The advantage of this new integral is that a single definition allows the integration of bounded or unbounded functions, in bounded or unbounded intervals. The present integral is different from the ordinary Riemann integral, where it is necessary to have the prior definition of bounded functions in bounded intervals.

Key-words. Riemann Integral, Improper Integral, Infinite Partitions

1 Introduction

The Riemann integral is commonly studied in initial Calculus courses. It has great historical, didactic and computational importance. Historic for being the first rigorous development, according to the criteria of modern mathematics; didactic because it is simple, with the concept of limit being the only prerequisite; and computational because many integrals can be calculated easily, via the Fundamental Theorem of Calculus. Its disadvantage is that it is defined only for bounded functions in bounded intervals. One can extend it to unbounded functions or unbounded intervals, which produces the improper integrals, but it is still necessary to have a prior definition of the integral for bounded functions in bounded intervals. The Darboux integral is interesting, it defines the integral by way of suprema and infima, but it is well known that the Darboux integral is equivalent to the Riemann integral [2] so no new results are brought by this device. The Lebesgue integral is a powerful tool for various fields of mathematics and science. It allows one to integrate a much larger number of functions than the Riemann integral and does not require a prior definition of the integral for bounded functions in bounded intervals because the same definition is used for bounded or unbounded functions in bounded or unbounded intervals. However, as is known, although every Riemann integrable, bounded function, defined in bounded intervals, is Lebesgue integrable, the same is not true for the Riemann improper integrals. There are functions whose Riemann improper integral is convergent

[1]tiago.reis@ifrj.edu.br

but which are not Lebesgue integrable. A classic example is the integral $\int_0^\infty \frac{\sin x}{x}$. The Henstock-Kurzweil integral is an effective generalisation of the Riemann integral. Every Riemann integrable function, both proper and improper, is Henstock-Kurzweil integrable. Moreover every Lebesgue integrable function, defined in intervals, is Henstock-Kurzweil integrable. However the Henstock-Kurzweil integral does not escape the need for a prior definition of the integral in bounded intervals, which is then extended to unbounded intervals [1].

This paper proposes a definition of the integral that is different from all of the above definitions. Like in Darboux, the integral is defined by way of upper and lower *sums* but the important difference is that here infinite partitions and infinite sums, that is, *series* instead of *finite sums* are used. The advantage of this new integral is that a single definition allows the integration of bounded or unbounded functions, in bounded or unbounded intervals. It is different from the Riemann integral, where it is necessary to have the prior definition for bounded functions in bounded intervals. In the ordinary Riemann integral, a partition of an interval is a finite set whose smallest element is the interval's lower end and the greatest element is the interval's upper end. In the integral proposed here, a partition of an interval is an infinite set whose elements converge to the interval's endpoints. This fact allows one to deal, *in the same way*, with both bounded and unbounded intervals. Then it is shown that every Riemann integrable function, both proper and improper, is integrable, in the sense established here, and both integrals have the same value. Furthermore for bounded functions, in bounded intervals, the Riemann integral and our integral are equivalent.

In the present paper some details of calculations are omitted every time that a statement starts with "it is possible to show that." Those details will be presented in a longer version of this work.

2 The Integral

In everything that follows, $\mathbb{R}^E$ denotes the set of extended real numbers, $[-\infty, \infty]$, provided with its usual topology.

Definition 2.1. *Let $a, b \in \mathbb{R}^E$ where $a < b$. A sequence $(x_n)_{n\in\mathbb{Z}}$ is called an* ***infinite partition*** *of $[a, b]$ if and only if $(x_n)_{n\in\mathbb{Z}}$ is strictly increasing and $\lim_{n\to-\infty} x_n = a$ and $\lim_{n\to\infty} x_n = b$.*

If $a, b \in \mathbb{R}^E$ with $a < b$, $f : [a, b] \to \mathbb{R}^E$ and $(x_n)_{n\in\mathbb{Z}}$ is an infinite partition of $[a, b]$, denote, for each $i \in \mathbb{Z}$, $\Delta x_i := x_i - x_{i-1}$, $m_i := \inf\{f(x);\ x \in [x_{i-1}, x_i]\}$, $M_i := \sup\{f(x);\ x \in [x_{i-1}, x_i]\}$, $I(f; (x_n)_{n\in\mathbb{Z}}) := \sum_{i=-\infty}^{\infty} m_i\Delta x_i$ if this series is convergent in $\mathbb{R}^E$ and $S(f; (x_n)_{n\in\mathbb{Z}}) := \sum_{i=-\infty}^{\infty} M_i\Delta x_i$ if this series is convergent in $\mathbb{R}^E$. Denote, by $\mathcal{P}_\infty(f; [a, b])$, the set of all the infinite partitions $(x_n)_{n\in\mathbb{Z}}$ of $[a, b]$ such that $\sum_{i=-\infty}^{\infty} m_i\Delta x_i$ and $\sum_{i=-\infty}^{\infty} M_i\Delta x_i$ are both convergent in $\mathbb{R}^E$.

Definition 2.2. *Let $a, b \in \mathbb{R}^E$ where $a < b$ and let $f : [a, b] \to \mathbb{R}^E$ be a function. The function f is said to be* ***integrable*** *in $[a, b]$ if and only if*

$$\sup\{I(f; P);\ P \in \mathcal{P}_\infty(f; [a, b])\} = \inf\{S(f; P);\ P \in \mathcal{P}_\infty(f; [a, b])\}. \tag{1}$$

And in this case the ***integral*** *of f in $[a,b]$ is defined by*

$$\int_a^b f := \sup\left\{I(f;P);\ P\in\mathcal{P}_\infty\big(f;[a,b]\big)\right\}. \tag{2}$$

Henceforth the integral, defined above, is called simply an **integral** and is denoted by $\int_a^b f$. The **Riemann integral** is denoted by $\underset{\mathcal{R}}{\int_a^b} f$.

Recall some definitions concerning the Riemann integral. If $a,b\in\mathbb{R}$ where $a\le b$, P is said to be a partition of $[a,b]$ if and only if $P=\{x_0,\dots,x_n\}$, for some $n\in\mathbb{N}$, and $a=x_0\le x_1\le\cdots\le x_{n-1}\le x_n=b$. The set of all partitions, P of $[a,b]$, is denoted by $\mathcal{P}\big([a,b]\big)$. If $a,b\in\mathbb{R}$ where $a\le b$, $f:[a,b]\to\mathbb{R}$ is a bounded function and $P\in\mathcal{P}\big([a,b]\big)$ then $\Delta x_i := x_i-x_{i-1}$; $m_i:=\inf\left\{f(x);\ x\in[x_{i-1},x_i]\right\}$; $M_i:=\sup\left\{f(x);\ x\in[x_{i-1},x_i]\right\}$ (the existence of m_i and M_i are ensured because f is bounded); $I(f;P):=\sum_{i=1}^n m_i\Delta x_i$ and $S(f;P):=\sum_{i=1}^n M_i\Delta x_i$. Remember also that $I(f;P)\le I(f;P\cup Q)\le S(f;P\cup Q)\le S(f;Q)$ for all $P,Q\in\mathcal{P}\big([a,b]\big)$. This ensures that $\left\{I(f;P);\ P\in\mathcal{P}\big([a,b]\big)\right\}$ is bounded from above and $\left\{S(f;P);\ P\in\mathcal{P}\big([a,b]\big)\right\}$ is bounded from below. The function f is Riemann integrable in $[a,b]$, if and only if $\sup\left\{I(f;P);\ P\in\mathcal{P}\big([a,b]\big)\right\}=\inf\left\{S(f;P);\ P\in\mathcal{P}\big([a,b]\big)\right\}$ and, in this case, the Riemann integral of f in $[a,b]$ is defined by

$$\underset{\mathcal{R}}{\int_a^b} f := \sup\left\{I(f;P);\ P\in\mathcal{P}\big([a,b]\big)\right\}=\inf\left\{S(f;P);\ P\in\mathcal{P}\big([a,b]\big)\right\}. \tag{3}$$

Proposition 2.1. *Let $a,b\in\mathbb{R}^E$ where $a<b$ and let $f:[a,b]\to\mathbb{R}^E$ be a function. It follows that*

$$I(f;P)\le S(f;Q), \tag{4}$$

for all $P,Q\in\mathcal{P}_\infty\big(f;[a,b]\big)$.

Proof. Let $a,b\in\mathbb{R}^E$ where $a<b$, let $f:[a,b]\to\mathbb{R}^E$ be a function and let $P,Q\in\mathcal{P}_\infty\big(f;[a,b]\big)$ be arbitrary. Say $P=(x_n)_{n\in\mathbb{Z}}$ and $Q=(y_n)_{n\in\mathbb{Z}}$. Define $z_0:=\min\{x_0,y_0\}$, denote $A:=\{x_n,y_n;\ n\in\mathbb{Z};$ such that $x_n>z_0$ and $y_n>z_0\}$ and, for each $n\ge 1$, define $z_n:=\min\left(A\setminus\{z_i;\ 0\le i\le n-1\}\right)$. Also denote $B:=\{x_n,y_n;\ n\in\mathbb{Z};$ such that $x_n<z_0$ and $y_n<z_0\}$ and, for each $n\le -1$, define $z_n:=\max\left(B\setminus\{z_i;\ n+1\le i\le 0\}\right)$. Notice that $(z_n)_{n\in\mathbb{Z}}\in\mathcal{P}_\infty\big(f;[a,b]\big)$.

Let $l\in\mathbb{N}$ such that $x_0=z_{l-1}$. For each $n\in\mathbb{N}$ there is $k_n\in\mathbb{N}$ such that $x_n=z_{k_n}$. Notice that $k_n<k_{n+1}$ for all $n\in\mathbb{N}$. Since $\left(\sum_{i=l}^{k_n} m_i\Delta z_i\right)_{n\in\mathbb{N}}$ is a subsequence of $\left(\sum_{i=l}^{k} m_i\Delta z_i\right)_{k\in\mathbb{N}}$, it follows that $\lim_{n\to\infty}\sum_{i=l}^{k_n} m_i\Delta z_i=\sum_{i=l}^{\infty} m_i\Delta z_i$. Notice also that $\sum_{i=1}^{n} m_i\Delta x_i\le\sum_{i=l}^{k_n} m_i\Delta z_i$ for all $n\in\mathbb{N}$ whence $\sum_{i=1}^{\infty} m_i\Delta x_i\le\lim_{n\to\infty}\sum_{i=l}^{k_n} m_i\Delta z_i=\sum_{i=l}^{\infty} m_i\Delta z_i$. In a similar way, $\sum_{i=-\infty}^{0} m_i\Delta x_i\le\sum_{i=-\infty}^{l-1} m_i\Delta z_i$. Thus $\sum_{i=-\infty}^{\infty} m_i\Delta x_i\le\sum_{i=-\infty}^{\infty} m_i\Delta z_i$. Now let $t\in\mathbb{N}$ such that $y_0=z_{t-1}$. For each $n\in\mathbb{N}$ there is $s_n\in\mathbb{N}$ such that $y_n=z_{s_n}$. Notice also that $\sum_{i=t}^{s_n} M_i\Delta z_i\le\sum_{i=1}^{n} M_i\Delta y_i$ for all $n\in\mathbb{N}$ whence

$\sum_{i=t}^{\infty} M_i\Delta z_i = \lim_{n\to\infty}\sum_{i=t}^{s_n} M_i\Delta z_i \le \sum_{i=1}^{\infty} M_i\Delta y_i$. In a similar way, $\sum_{i=-\infty}^{t-1} M_i\Delta z_i \le \sum_{i=-\infty}^{0} M_i\Delta y_i$. Thus $\sum_{i=-\infty}^{\infty} M_i\Delta z_i \le \sum_{i=-\infty}^{\infty} M_i\Delta y_i$. Thus

$$I(f;P) = \sum_{i=-\infty}^{\infty} m_i\Delta x_i \le \sum_{i=-\infty}^{\infty} m_i\Delta z_i \le \sum_{i=-\infty}^{\infty} M_i\Delta z_i \le \sum_{i=-\infty}^{\infty} M_i\Delta y_i = S(f;Q). \quad (5)$$

□

Corollary 2.1. *Let $a, b \in \mathbb{R}^E$ where $a < b$ and let $f : [a,b] \to \mathbb{R}^E$ be a function. It follows that*

$$\sup\left\{I(f;P);\ P \in \mathcal{P}_\infty\big(f;[a,b]\big)\right\} \le \inf\left\{S(f;P);\ P \in \mathcal{P}_\infty\big(f;[a,b]\big)\right\}. \quad (6)$$

Lemma 2.1. *Let $a \in \mathbb{R}$, $b \in \mathbb{R}^E$ where $a < b$ and let $f : [a,b] \to \mathbb{R}^E$ be a function with $f(x) \in \mathbb{R}$ for all $x \in [a,b)$ and where f is Riemann integrable on every closed subinterval of $[a,b)$. If the improper Riemann integral $\mathcal{R}\int_a^b f$ is convergent then, for each positive $\varepsilon \in \mathbb{R}$, there is a $P_\varepsilon \in \mathcal{P}_\infty\big(f;[a,b]\big)$ such that*

$$\mathcal{R}\int_a^b f - \varepsilon \le I(f;P_\varepsilon) \le S(f;P_\varepsilon) \le \mathcal{R}\int_a^b f + \varepsilon. \quad (7)$$

Proof. Let $a \in \mathbb{R}$, $b \in \mathbb{R}^E$ where $a < b$ and let $f : [a,b] \to \mathbb{R}^E$ be a function with $f(x) \in \mathbb{R}$ for all $x \in [a,b)$ and where f is Riemann integrable on every closed subinterval of $[a,b)$ such that the improper Riemann integral $\mathcal{R}\int_a^b f$ is convergent.

Let $b = \infty$. The case $b \in \mathbb{R}$ is analogous. Say $\mathcal{R}\int_a^\infty f = L \in \mathbb{R}$. Let $\varepsilon \in \mathbb{R}$ be arbitrary positive. It is possible to show that there are $(N_n)_{n\in\mathbb{N}} \subset \mathbb{R}$, $(k_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ and $(x_n)_{n\ge 0} \subset \mathbb{R}$ strictly increasing sequences such that $x_0 = a$, $N_n \ge n$, $x_{k_n} = N_n$, $\lim_{n\to\infty} x_n = \infty$ and

$$L - \frac{1}{n} - \varepsilon < \mathcal{R}\int_a^{N_n} f - \varepsilon < \sum_{i=1}^{k_n} m_i\Delta x_i \le \sum_{i=1}^{k_n} M_i\Delta x_i < \mathcal{R}\int_a^{N_n} f + \varepsilon < L + \frac{1}{n} + \varepsilon. \quad (8)$$

From (8) it follows that $\left(\sum_{i=1}^{k_n} m_i\Delta x_i\right)_{n\in\mathbb{N}}$ and $\left(\sum_{i=1}^{k_n} M_i\Delta x_i\right)_{n\in\mathbb{N}}$ are bounded sequences of real numbers. Thus there are $(k_s)_{s\in\mathbb{N}}$ and $(k_r)_{r\in\mathbb{N}}$ which are subsequences of $(k_n)_{n\in\mathbb{N}}$ such that $\left(\sum_{i=1}^{k_s} m_i\Delta x_i\right)_{s\in\mathbb{N}}$ and $\left(\sum_{i=1}^{k_r} M_i\Delta x_i\right)_{r\in\mathbb{N}}$ are convergent sequences and, furthermore,

$$L - \varepsilon \le \lim_{s\to\infty}\sum_{i=1}^{k_s} m_i\Delta x_i \le \lim_{r\to\infty}\sum_{i=1}^{k_r} M_i\Delta x_i \le L + \varepsilon. \quad (9)$$

It is possible to show that $\sum_{i=1}^{\infty} m_i \Delta x_i$ and $\sum_{i=1}^{\infty} M_i \Delta x_i$ are convergent and $\sum_{i=1}^{\infty} m_i \Delta x_i = \lim_{s\to\infty} \sum_{i=1}^{k_s} m_i \Delta x_i$ and $\sum_{i=1}^{\infty} M_i \Delta x_i = \lim_{r\to\infty} \sum_{i=1}^{k_r} M_i \Delta x_i$. Thus

$$L - \varepsilon \leq \sum_{i=1}^{\infty} m_i \Delta x_i \leq \sum_{i=1}^{\infty} M_i \Delta x_i \leq L + \varepsilon. \tag{10}$$

Let $(z_n)_{n\in\mathbb{N}}$ be a strictly decreasing sequence, convergent to a. Then there are $v \in \mathbb{N}$ such that $z_v < x_1$. Define $w_i := z_{v-i}$, for all $i \leq 0$, and define $w_i = x_i$, for all $i \geq 1$. It is possible to show that $P_\varepsilon := (w_n)_{n\in\mathbb{Z}} \in \mathcal{P}_\infty\big(f;[a,\infty]\big)$ and

$$\underset{\mathcal{R}}{\int_a^\infty} f - \varepsilon \leq I(f;P_\varepsilon) \leq S(f;P_\varepsilon) \leq \underset{\mathcal{R}}{\int_a^\infty} f + \varepsilon. \tag{11}$$

□

Theorem 2.1. *Let $a, b \in \mathbb{R}^E$ where $a < b$ and let $f : [a,b] \to \mathbb{R}^E$ be a function. If f is Riemann integrable, either as a proper integral or as an improper integral, then f is integrable and*

$$\int_a^b f = \underset{\mathcal{R}}{\int_a^b} f. \tag{12}$$

Proof. Let $a, b \in \mathbb{R}^E$ where $a < b$ and let $f : [a,b] \to \mathbb{R}^E$ be a function.

i) If $a, b \in \mathbb{R}$, $f\big([a,b]\big) \subset \mathbb{R}$ and f is bounded and Riemann integrable then, by (3),

$$\sup\left\{I(f;P);\ P \in \mathcal{P}\big([a,b]\big)\right\} = \underset{\mathcal{R}}{\int_a^b} f = \inf\left\{S(f;P);\ P \in \mathcal{P}\big([a,b]\big)\right\}. \tag{13}$$

It is possible to show that $\sup\left\{I(f;P);\ P \in \mathcal{P}\big([a,b]\big)\right\} \leq \sup\left\{I\big(f;P);\ P \in \mathcal{P}_\infty\big(f;[a,b]\big)\right\}$ and $\inf\left\{S(f;P);\ P \in \mathcal{P}_\infty\big(f;[a,b]\big)\right\} \leq \inf\left\{S(f;P);\ P \in \mathcal{P}\big([a,b]\big)\right\}$. From Corollary 2.1,

$$\begin{aligned} \sup\left\{I(f;P);\ P \in \mathcal{P}\big([a,b]\big)\right\} &\leq \sup\left\{I(f;P);\ P \in \mathcal{P}_\infty\big(f;[a,b]\big)\right\} \\ &\leq \inf\left\{S(f;P);\ P \in \mathcal{P}_\infty\big(f;[a,b]\big)\right\} \\ &\leq \inf\left\{S(f;P);\ P \in \mathcal{P}\big([a,b]\big)\right\}. \end{aligned} \tag{14}$$

From (13) it follows that f is integrable and $\int_a^b f = \underset{\mathcal{R}}{\int_a^b} f$.

ii) If $a \in \mathbb{R}$, $f\big([a,b)\big) \subset \mathbb{R}$, f is Riemann integrable on every closed subinterval of $[a,b)$ and the improper integral $\underset{\mathcal{R}}{\int_a^b} f$ is convergent. From Lemma 2.1, for each positive $\varepsilon \in \mathbb{R}$ there are $P_\varepsilon \in \mathcal{P}_\infty\big(f;[a,b]\big)$ such that

$$\underset{\mathcal{R}}{\int_a^b} f - \varepsilon \leq I(f;P_\varepsilon), \tag{15}$$

whence

$$\underset{\mathcal{R}}{\int_a^b} f \leq \sup\left\{I(f;P);\ P \in \mathcal{P}_\infty\big(f;[a,b]\big)\right\}. \tag{16}$$

The inequality

$$\inf\left\{S(f;P);\ P \in \mathcal{P}_\infty\big(f;[a,b]\big)\right\} \leq \underset{\mathcal{R}}{\int_a^b} f \tag{17}$$

is obtained in a similar way. By Corollary 2.1,

$$\sup\left\{I(f;P);\ P \in \mathcal{P}_\infty\big(f;[a,b]\big)\right\} \leq \inf\left\{S(f;P);\ P \in \mathcal{P}_\infty\big(f;[a,b]\big)\right\}. \tag{18}$$

Thus

$$\sup\left\{I(f;P);\ P \in \mathcal{P}_\infty\big(f;[a,b]\big)\right\} = \inf\left\{S(f;P);\ P \in \mathcal{P}_\infty\big(f;[a,b]\big)\right\} = \underset{\mathcal{R}}{\int_a^b} f. \tag{19}$$

iii) If $b \in \mathbb{R}$ and $f\big((a,b]\big) \subset \mathbb{R}$ and f is Riemann integrable on every closed subinterval of $(a,b]$ such that the improper integral $\underset{\mathcal{R}}{\int_a^b} f$ is convergent then the proof follows in a similar way to case (ii) with a similar lemma to Lemma 2.1.

iv) If $f\big((a,b)\big) \subset \mathbb{R}$, f is Riemann integrable on every closed subinterval of (a,b) and the improper integral $\underset{\mathcal{R}}{\int_a^b} f$ is convergent then the proof follows in a similar way to case (ii) with a similar lemma to the Lemma 2.1. $\square$

Theorem 2.2. *Let $a,b \in \mathbb{R}$ where $a < b$ and let $f : [a,b] \to \mathbb{R}$ be a bounded function. It follows that f is integrable if and only if f is Riemann integrable. And in this case*

$$\int_a^b f = \underset{\mathcal{R}}{\int_a^b} f. \tag{20}$$

Proof. Let $a,b \in \mathbb{R}$ where $a < b$ and let $f : [a,b] \to \mathbb{R}$ be a bounded function. That Riemann integrability implies integrability is already seen in Theorem 2.1. Now suppose that f is integrable. It follows that

$$\sup\left\{I(f;P);\ P \in \mathcal{P}_\infty\big(f;[a,b]\big)\right\} = \inf\left\{S(f;P);\ P \in \mathcal{P}_\infty\big(f;[a,b]\big)\right\}. \tag{21}$$

Denote $L := \sup\left\{I(f;P);\ P \in \mathcal{P}_\infty\big(f;[a,b]\big)\right\}$. Let there be an arbitrary $n \in \mathbb{N}$. Then there is a $P_n \in \mathcal{P}_\infty\big(f;[a,b]\big)$ such that $L - \frac{1}{n} < I(f;P_n)$. Say $P_n = (x_i)_{i\in\mathbb{Z}}$. Since f is a bounded function, there is positive $A \in \mathbb{R}$ such that $-A \leq f(x)$ for all $x \in [a,b]$. Let $\varepsilon := \sum_{i=-\infty}^{\infty} m_i \Delta x_i - L + \frac{1}{n} = I(f;P_n) - L + \frac{1}{n} > 0$. Then there are $k \in \mathbb{N}$ such that $x_{-k} < a + \frac{\varepsilon}{3A}$ and $x_k > b - \frac{\varepsilon}{3A}$ and $s \in \mathbb{N}$ such that $\sum_{i=-\infty}^{\infty} m_i \Delta x_i - \frac{\varepsilon}{3} < \sum_{i=-j}^{j} m_i \Delta x_i$ for all $j \geq s$. Take $l = \max\{s,k\}$. Thus $\sum_{i=-\infty}^{\infty} m_i \Delta x_i - \frac{\varepsilon}{3} < \sum_{i=-l}^{l} m_i \Delta x_i$, $x_{-l-1} < a + \frac{\varepsilon}{3A}$

whence $-\frac{\varepsilon}{3} < -A(x_{-l-1}-a) \leq \inf f\big([a,x_{-l-1}]\big)(x_{-l-1}-a)$ and $x_l > b-\frac{\varepsilon}{3A}$ whence $-\frac{\varepsilon}{3} < -A(b-x_l) \leq \inf f\big([x_l,b]\big)(b-x_l)$. Thus, it follows that $L-\frac{1}{n} = \sum_{i=-\infty}^{\infty} m_i\Delta x_i - \varepsilon = -\frac{\varepsilon}{3} + \sum_{i=-\infty}^{\infty} m_i\Delta x_i - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} < \inf f\big([a,x_{-l-1}]\big)(x_{-l-1}-a) + \sum_{i=-l}^{l} m_i\Delta x_i + \inf f\big([x_l,b]\big)(b-x_l)$. Denoting $y_0 := a$, $y_i := x_{-l-2+i}$ for all $i \in \{1,\ldots,2l+2\}$ and $y_{2l+3} := b$, it follows that $L-\frac{1}{n} < \inf f\big([a,x_{-l-1}]\big)(x_{-l-1}-a) + \sum_{i=-l}^{l} m_i\Delta x_i + \inf f\big([x_l,b]\big)(b-x_l) = \sum_{i=1}^{2l+3} m_i\Delta y_i$. Denoting $Q_n := \{y_0,\ldots,y_{2l+3}\}$ it follows that $Q_n \in \mathcal{P}\big([a,b]\big)$ and $L-\frac{1}{n} < I(f;Q_n) \leq \sup\big\{I(f;P);\ P \in \mathcal{P}\big([a,b]\big)\big\}$. Thus

$$\sup\big\{I(f;P);\ P \in \mathcal{P}_\infty\big(f;[a,b]\big)\big\} = L \leq \sup\big\{I(f;P);\ P \in \mathcal{P}\big([a,b]\big)\big\}. \tag{22}$$

In a similar way it follows that

$$\inf\big\{I(f;P);\ P \in \mathcal{P}\big([a,b]\big)\big\} \leq \inf\big\{I(f;P);\ P \in \mathcal{P}_\infty\big(f;[a,b]\big)\big\}. \tag{23}$$

Thus f is Riemann integrable and

$$\underset{\mathcal{R}}{\int_a^b} f = \int_a^b f. \tag{24}$$

□

3 Conclusion

This paper defines an integral by way of upper and lower infinite sums, that is, upper and lower series. A partition of an interval is an infinite set whose elements converge to the interval's endpoints. The advantage of this new integral is that a single definition allows the integration of bounded or unbounded functions, in bounded or unbounded intervals. It is different from the Riemann integral, where it is necessary to have the prior definition for bounded functions in bounded intervals. It is shown that every Riemann integrable function, both proper and improper, is integrable, in the sense established here, and both integrals have the same value. Furthermore for bounded functions, in bounded intervals, the Riemann integral and our integral are equivalent.

Acknowledgement

I thank James A. D. W. Anderson for revising this paper and for the conversations that led to it.

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