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Global stability of fractional SIR epidemic model

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Abstract. In this work, we prove the global stability of endemic and free disease equilibrium points of the Fractional SIR model using comparison theory of fractional differential inequality and fractional La-Salle invariance principle for fractional differential equations.

Keywords. Caputo fractional derivative, LaSalle invariance principle, SIR disease model.

1 Introduction

The qualitative behavior of ordinary differential equations systems which describe diseases has been studied for a long time and is an important issue in the real world. The first model that can be used to interpret the disease characteristic of epidemics is a susceptible-infected-recuperated model SIR, that was developed by Kermack and McKendrick [4]. Several extensions of this model have been used to describe diseases in the Literature. Recently, fractional derivatives have been used to generalized models describe epidemic disease [3, 8, 9, 11], the motivation is due the fractional-order differential equation systems reproduce with more efficacy the reality [1]. In this work, we generalize the results of the global stability for SIR models for a fractional model describing the dynamics of SIR models.

2 Preliminaries

For a long time, there have been several definitions that fit the concept of fractional derivatives [2, 10]. In this paper we study the Caputo fractional derivative. Firstly, we introduce the definition of Riemann-Liouville fractional integral $J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds$, where $\alpha > 0$, $f \in L^1(\mathbb{R}^+)$ and $\Gamma(\cdot)$ is the Gamma function. The Caputo fractional derivative is given by $D_t^{\alpha} f(t) = J^{n-\alpha}[f^n(t)]$, $n-1 < \alpha < n$.

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The Laplace transform of the Caputo fractional derivative is given by

$$\mathcal{L}[D_t^\alpha f(t)] = s^\alpha \mathcal{L}[f(t)] - \sum_{k=0}^{n-1} f^k(0) s^{\alpha-k-1}. \tag{1}$$

Next, we present the Mittag-Leffer function defined by the following infinite power series

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}. \tag{2}$$

The Mittag-Leffer functions satisfy the equality

$$E_{\alpha,\beta}(z) = zE_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)}. \tag{3}$$

The Laplace transform of the functions

$$\mathcal{L}[t^{\beta-1}E_{\alpha,\beta}(-at^\alpha)] = \frac{s^{\alpha-\beta}}{s^\alpha + a}. \tag{4}$$

Next we introduce definitions and results about the comparison theory of fractional differential equations.

Definition 2.1. *A function f is Hölder continuous if there are nonnegative constants C, ν such that $||f(x)-f(y)|| \leq C||x-y||^\nu$, for all x, y in the domain of f and ν is the Hölder exponent. We represent the space of Hölder continuous functions by $C^{0,\nu}$.*

We develop a generalized inequality, wherein the underlying comparison system is a vector fractional-order system. A nonnegative (respectively, positive) vector v means that every component of v is nonnegative (respectively, positive). We denote a nonnegative (respectively, positive) vector with $0 \leq\leq v$ (respectively $0 << v$).

Consider the fractional-order system

$$D_C^\theta u(t) \quad = \quad f(t,u), \quad u(0) = u_0, \tag{5}$$

where, $D_C^\theta u(t) = (D_C^\theta u_1(t), D_C^\theta u_2(t), \dots, D_C^\theta u_m(t))^T$, $0 < \theta < 1, u(t) \in \mathcal{M} \subset \mathbb{R}^m, t \in [0,T)(T \leq +\infty), \mathcal{M}$ is an open set, $0 \in \mathcal{M}$, and $f : [0,T) \times \mathcal{M} \to \mathbb{R}^m$ is continuous in t and satisfies the Lipschitz condition $\| f(t,u') - f(t,u'') \| \leq L \| u' - u'' \|, \quad t \in [0,T)$, for all $u', u'' \in \Omega \subset \mathcal{M}$, where $L > 0$ is a Lipschitz constant.

Theorem 2.1. *[12] Let $u(t), t \in [0,T)$, be the solution of system (5). If there exists a vector function $v = (v_1, v_2, \dots, v_m)^T : [0,T) \to \mathcal{M}$ such that $v_i \in C^{0,\nu}, \theta < \nu < 1, i = 1, \dots, m$ and $D_C^\theta v(t) \leq\leq f(t, v(t)), \quad t \in [0,T)$. If $v(0) \leq\leq u_0, u_0 \in \mathcal{M}$, then $v(t) \leq\leq u(t), t \in [0,T)$.*

Let $f : D \to \mathbb{R}^n, D \subseteq \mathbb{R}^n$, next, we study the qualitative behavior of solutions of the fractional order systems

$$\begin{cases} D_t^\alpha x(t) &= f(x(t)), \\ x(0) &= x_0. \end{cases} \tag{6}$$

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Definition 2.2. *We say that e is an equilibrium point for (6), if and only if,* $f(e) = 0$.

Remark 2.1. *When* $\alpha \in (0,1)$, *the fractional system* $D_t^{\alpha}x(t) = f(x)$ *has the same equilibrium points as the system* $x'(t) = f(x)$.

Definition 2.3. *The equilibrium point e of the autonomous system (6) is said to be stable if for all* $\epsilon > 0$, *exists* $\delta > 0$ *such that if* $\| x_0 - e \| < \delta$ *then* $\| x(t) - e \| < \epsilon, t \geq 0$; *the equilibrium point e of the autonomous system (6) is said asymptotically stable if it is stable and* $\lim_{t\to\infty} x(t) = e$.

The next result establishes the stability of the fractional linear system similar to the theory of ordinary differential equation.

Theorem 2.2. *[7] Let* $A \in M_{m\times m}(\mathbb{R})$, *the origin of the system* $D_C^{\theta}x(t) = Ax(t)$ *is asymptotically stable if and only if* $|arg(\lambda_i)| > \frac{\theta\pi}{2}$ *is satisfied for all eigenvalues of the matrix A. Moreover, this system is stable if and only if,* $|arg(\lambda_i)| \geq \frac{\theta\pi}{2}$, *is satisfied for all eigenvalues of the matrix A, and the eigenvalues satisfying* $|arg(\lambda_i)| = \frac{\theta\pi}{2}$, *have geometric multiplicity equal to one.*

The next result by [6] defined the concept of fractional dynamical systems in Caputo sense for the fractional systems (6).

Theorem 2.3. *[6] Let* $f(\cdot)$ *be a continuous function and* $x(\cdot)$ *be the continuous solution of (6), then there exists a* ϕ_t *which satisfies the following properties:*

(a3) $\phi_0 = Id$.

(b3) $\phi_{t+s} = \phi_t \circ \theta_t \circ \phi_s, s,t \in \mathbb{R}^+$, *where* θ_t *is a linear map satisfying:* $\theta_t \circ \phi_s(x_0) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^s (t+s-\tau)^{\alpha-1} f(\phi_\tau)d\tau, \quad t \geq 0$, *and if* $s = 0, \theta_t(x_0) = x_0$.

(c3) $(t, x_0) \to \phi_t(x_0)$ *gives a continuous map from* $\mathbb{R}^+ \times \Omega \to \Omega$.

Definition 2.4. ϕ_t, *which satisfies* $(a3)-(c3)$, *is called a fractional flow in Caputo sense, and* $\{\mathbb{R}^+, \Omega, \phi_t\}$ *is a fractional dynamical system in Caputo sense.*

Definition 2.5. *Let* $x(\cdot)$ *be a solution of (6), a point p is said to be a positive limit point of* $x(\cdot)$ *if there is a sequence* $\{t_n\}$, *with* $t_n \to \infty$ *as* $n \to \infty$, *such that* $x(t_n) \to p$ *as* $n \to \infty$. *The set of all positive limit points of* $x(\cdot)$ *is called the positive limit set of* $x(\cdot)$, *we denote this set by* $L_\alpha^+(x)$.

Definition 2.6. *A set M is said to be an invariant set with respect to (6) if* $x(0) \in M$ *implies* $x(t) \in M$, *for all* $t \in \mathbb{R}$. *A set M is said to be a positively invariant set if* $x(0) \in M$ *implies* $x(t) \in M$, *for all* $t \geq 0$. *We also say that* $x(\cdot)$ *approaches a set M as t approaches infinity, if for each* $\epsilon > 0$ *there is* $T > 0$ *such that* $dist(x(t), M)) < \epsilon$, *for all* $t > T$.

We now introduce the Fractional LaSalle's invariance principle.

Theorem 2.4. *[9] Let* $\Omega \subset D$ *be a positively invariant set with respect to (6). Let* $V : D \to \mathbb{R}$ *be a continuously differentiable function such that* $V(x) > 0$ *and* $D_t^{\alpha}V(x(t)) \leq 0$ *in* Ω *for* $x(t)$ *solutions of the system (6). Let E be the set of all points in* Ω *where* $D_t^{\alpha}V(x) = 0$. *Let M be the largest invariant set in E, then every bounded solution starting in* Ω *approaches M as* $t \to \infty$.

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3 Global stability of fractional SIR model

In this section we briefly introduce the mathematical SIR model. We divide the population into three subpopulations of epidemiological significance, the Susceptible class: individuals who can incur the disease but are not infective; Infective class: individuals who are transmitting the disease to others; and, Removed class: individuals who are removed form the susceptible-infective interaction by immunity or isolation. The fraction of the total population in these classes is $\overline{S}(t), \overline{I}(t)$ and $\overline{R}(t)$. The population has constant size N. The death removal rate is denoted by μ. The average lifetime is $1/\mu$. The average number of contacts per infective, per day, which result in infection, is denoted by λ. The average fraction of susceptible infected by the infective class is $\lambda\overline{SI}$. Individuals recover from the infective class at a per capita constant rate γ. Then new SIR model with Caputo derivative for $\alpha \in (0,1)$ is given by

$$\left\{\begin{array}{rcl} D_t^{\alpha}\overline{S}(t) & = & \mu - \lambda\overline{SI} - \mu\overline{S}, \\ D_t^{\alpha}\overline{I}(t) & = & \lambda\overline{SI} - \mu\overline{I}, \\ D_t^{\alpha}\overline{R}(t) & = & \gamma\overline{I} - \mu\overline{R}. \end{array}\right. \tag{7}$$

Making the changes $S(t) = \frac{\overline{S}(t)}{N}, I(t) = \frac{\overline{I}(t)}{N}$ and $R(t) = \frac{\overline{R}(t)}{N}$, we obtain $S(t)+I(t)+R(t) = 1$. From above, we obtain the following equivalent fractional system that describes the dynamics of SIR model

$$\left\{\begin{array}{rcl} D_t^{\alpha}S(t) & = & \mu - \lambda SI - \mu S, \\ D_t^{\alpha}I(t) & = & \lambda SI - (\mu+\gamma)I. \end{array}\right. \tag{8}$$

In this system, we obtain the equilibrium points E_0 and E^* where

$$E_0 = (1,0) \text{ and } E^* = (S^*, I^*) = \left(\frac{1}{R_0}, \frac{\mu}{\lambda}\left(R_0 - 1\right)\right),$$

where where the parameter $R_0 = \frac{\lambda}{\gamma+\mu}$, is called the basic reproduction number.

In the next result we prove the existence of positive invariant set for the fractional differential system (8).

Theorem 3.1. *Let (S,I) be the solution of the fractional system (8) with the initial condition $(S(0), I(0))$ in the compact set*

$$D = \{(S,I) \in \mathbb{R}_+^2 : S \geq 0, I \geq 0 \text{ and } S + I \leq 1\}.$$

Then, D is a positively invariant set.

Proof: Using the same argument of Proposition 10 in [8], we proove the axis $S = 0$ and $I = 0$ are solutions and invariants sets. By existence and unicity, see [5], we obtain that solutions of (8) do not cross the axis S and I. If $S(0) + I(0) \leq 1$ from two equations of (8) we obtain

$$D_t^{\alpha}(S+I) \leq \mu - \mu(S+I).$$

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Applying the Laplace transform in the previous inequality, we obtain

$$\lambda^{\alpha}\mathcal{L}(S+I)-\lambda^{\alpha-1}(S(0)+I(0)) \quad \leq \quad \frac{\mu}{\lambda}-\mu\mathcal{L}(S+I),$$

we can written as

$$\mathcal{L}(S+I) \quad \leq \quad \mu\frac{\lambda^{\alpha-(1+\alpha)}}{\lambda^{\alpha}+\mu}+\frac{\lambda^{\alpha-1}}{\lambda^{\alpha}+\mu}(S(0)+I(0))$$

From (4) we infer

$$S(t)+I(t) \quad \leq \quad \mu t^{\alpha}E_{\alpha,\alpha+1}(-\mu t^{\alpha})+E_{\alpha,1}(-\mu t^{\alpha})(S(0)+I(0)),$$

by $S(0)+I(0)\leq 1$, and (3) we obtain $S(t)+I(t)\leq \mu t^{\alpha}E_{\alpha,\alpha+1}(-\mu t^{\alpha})+E_{\alpha,1}(-\mu t^{\alpha})=1$, this implies that D is a positively invariant set. This concludes the proof.

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Theorem 3.2. *If $R_0<1$, then the disease equilibrium E_0 is globally asymptotically stable on D.*

Proof: From the change $X(t)=S(t)-1$, the system (8) is equivalent to the system

$$\begin{cases} D_t^{\alpha}X(t) &= -\mu X-\lambda XI-\lambda I,\\ D_t^{\alpha}I(t) &= \lambda XI-(\lambda+\mu+\gamma)I. \end{cases} \tag{9}$$

By $X(t)\leq 1$ and $I(t)\leq 1$ it is easy to see that

$$\begin{aligned} -\mu X-\lambda XI-\lambda I &\leq -\mu X-\lambda I,\\ \lambda XI-(\lambda+\mu+\gamma)I &\leq \lambda I-(\mu+\gamma)I. \end{aligned}$$

From the above, it follows the solutions of the system (9) satisfies the differential inequality

$$\begin{cases} D_t^{\alpha}X &\leq -\mu X-\lambda I,\\ D_t^{\alpha}I &\leq \lambda I-(\mu+\gamma)I. \end{cases} \tag{10}$$

Moreover, motivated by (10), let $(x(t),i(t))$ be the solution of fractional linear system

$$\begin{cases} D_t^{\alpha}x &= -\mu x-\lambda i,,\\ D_t^{\alpha}i &= \lambda i-(\mu+\gamma)i.. \end{cases} \tag{11}$$

with initial conditions $(x(0),i(0))=(x_0,i_0)\in D$. The eigenvalues of the system (11) are given by

$$\begin{vmatrix} -\mu-\xi & -\lambda\\ 0 & \lambda-(\gamma+\mu)-\xi \end{vmatrix}=(\mu-\xi)(\lambda-(\gamma+\mu)-\xi)=0.$$

It is easy to see that $\xi=-\mu$ and $\xi=\lambda-(\gamma+\mu)=(\gamma+\mu)(R_0-1)$.

If $R_0<1$, we infer that all the eigenvalues are negatives, thus $|arg(\lambda_i)|=\pi,\ i=1,2$, from Theorem 2.2, we can conclude that $\lim\limits_{t\to\infty}x(t)=0$, and $\lim_{t\to\infty}i(t)=0$. From previous discussion and comparison principle given by Theorem 2.1, we obtain

$$(X(t),I(t))\leq\leq(x(t),i(t)),$$

this implies $\lim\limits_{t\to\infty}(X(t),I(t))=(0,0)$. It follows that $(S(t),I(t))$ converge to the disease-free equilibrium point E_0. This finishes the proof.

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The next Lemma is a direct consequence of [11, Lemma 3.1] and so we omit its proofs.

Lemma 3.1. *Let $x(t) \in \mathbb{R}^+$ be a continuous functions. Then, for any time $t \leq 0$*

$$D_t^\alpha \left[(x(t) - a \ln x(t)\right] \leq \left(1 - \frac{a}{x(t)}\right) D_t^\alpha x(t), \quad a \in \mathbb{R}^+. \tag{12}$$

Theorem 3.3. *If $R_0 > 1$, then the positive endemic equilibrium state E^* of system (8) exists and is globally asymptotically stable on $D_+ = \{(S, I) \in D : S > 0, I > 0\}$.*

Proof: Let the Lyapunov function candidate $U : D_+ \to \mathbb{R}$, and $U(S, I) = (S - S^* \ln S) + (I - I^* \ln I)$. We have by Lemma 3.1 that

$$\begin{aligned} D_t^\alpha U(S, I) &\leq \frac{(S - S^*)}{S} D_t^\alpha S(t) + \frac{(I - I^*)}{I} D_t^\alpha I(t) \\ &= \frac{(S - S^*)}{S}(\mu - \lambda S I - \mu S) + \frac{(I - I^*)}{I}(\lambda S I - (\mu + \gamma) I). \end{aligned}$$

From the equations at equilibrium $-\mu = \lambda I^* - \frac{\mu}{S^*}$ and $-(\mu + \gamma) = -\lambda S^*$ we have

$$\begin{aligned} D_t^\alpha U(S, I) &\leq -\lambda(S - S^*)(I - I^*) - \mu \frac{(S - S^*)^2}{S S^*} + \lambda(I - I^*)(S - S^*) \\ &= -\mu \frac{(S - S^*)^2}{S S^*} < 0. \end{aligned}$$

Using the Theorem 2.4, the limit set of each solution is contained in the largest invariant set in $E = \{(S, I) \in D : D_t^\alpha U(S, I) = 0\}$, we proove that $E \subseteq \{(S, I) \in D : S = S^*\}$. By contraction, if there exists (S, I) such that $D_t^\alpha U(S, I) = 0$ and $S \neq S^*$, we infer $0 = D_t^\alpha U(S, I) \leq -\mu \frac{(S-S^*)^2}{SS^*} < 0$, this is absurd. This implies that $E \subseteq \{(S, I) \in \mathbb{R}_+^2 \in D : S = S^*, I = I^*\} = \{E^*\}$. This shown that $E = E^*$ and the endemic equilibrium point E^* is globally asymptotically stable on D_+. This concludes the proof.

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4 Conclusions

A new result for global asymptotic stability of disease-free equilibrium using comparison theory of fractional differential equations and the global stability of endemic equilibrium point using the Lasalle invariance principle for fractional differential is presented. The proof shown here should be used as a guide in the study of equilibrium conditions in similar problems, such as tuberculosis, or toxoplasmosis and others diseases.

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References

[1] K. Diethelm. A fractional calculus based model for the simulation of an outbreak of dengue fever. *Nonlinear Dynamics*, 71(4), 613-619, 2013.

[2] K. Diethelm. *The analysis of fractional differential equations:* An application-oriented exposition using operators of Caputo type (Vol. 2004).

[3] G. González-Parra, A. J. Arenas, B. M. Chen-Charpentier. A fractional order epidemic model for the simulation of outbreaks of influenza A (H1N1). *Mathematical Methods in the Applied Sciences*, 37(15), 2218-2226, 2014.

[4] W. O. Kermack, A. G. McKendrick. A contribution to the mathematical theory of epidemics. In *Proceedings of the Royal Society of London*: Mathematical, Physical and Engineering Sciences 115, (772), 700-721, 1927.

[5] W. Lin, Global existence theory and chaos control of fractional differential equations, *Journal of Mathematical Analysis and Applications* 332, 709-726, 2007.

[6] C. Li, Y. Ma. Fractional dynamical system and its linearization theorem. *Nonlinear Dynamics*, 71(4), 621-633, 2013.

[7] D. Matignon. Stability results for fractional differential equations with applications to control processing. In *Computational engineering in systems applications*, v. 2, 963-968, 1996.

[8] J. P. C dos Santos, L. C. Cardoso, E. Monteiro, N. T. Lemes. A Fractional-Order Epidemic Model for Bovine Babesiosis Disease and Tick Populations. *Abstract and Applied Analysis*, 1-10, 2015.

[9] J. P. C dos Santos, E. Monteiro, J. C. Valverde. *Fractional LaSalle invariance principe and applications on global stability of Dengue model*, Preprint.

[10] I. Podlubny. *Fractional derivatives: history, theory, application.* Utah State University, Logan, 2005.

[11] C. Vargas-De-Léon. Volterra Lyapunov functions for fractional-order epidemic sytems, *Comm. Nonlinear Csi. Numer Simulat*, 24, 75-85, 2004.

[12] Z. Wang, D. Yang, T. Ma, N. Sun. Stability analysis for nonlinear fractional-order systems based on comparison principle. *Nonlinear Dynamics*, 75(1-2), 387-402, 2014.

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