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Chaotic sliding dynamics for planar piecewise affine maps

Claudio A. Buzzi[1]
Instituto de Biociências, Letras e Ciências Exatas, IBILCE-UNESP, S. J. Rio Preto, SP
João C. Medrado[2]
Instituto de Matemática e Estatística, IME-UFG, Goiânia, GO
Paulo R. da Silva[3]
Instituto de Biociências, Letras e Ciências Exatas, IBILCE-UNESP, S. J. Rio Preto, SP

Abstract. In this work we adapt the theory of piecewise differential systems to the world of piecewise affine maps. Through this adaptation it is possible to define a sliding map in the sense of Filippov. We prove that there exists a choice for the pieces of the piecewise affine map, such that the associate sliding map possesses chaotic dynamics.

Keywords. Dynamical Systems, Piecewise Affine Maps, Filippov Systems, Chaos.

1 Introduction

Basically we can classify the theory of dynamical systems in two classes: continuous and discrete. Roughly speaking, the continuous ones are those where the dynamics is given by flows of vector fields; and the discrete ones are those where the dynamics is given by maps.

Piecewise smooth differential systems have frequently appeared in different fields of science such as control theory, mechanical engineering with impact and dry frictions, electronic circuits with switches and so on, see e.g. [2,3,9,10,13] and the references therein. Theory on dynamics of piecewise smooth differential systems has had advance since the pioneering works of Andronov [1] and Filippov [7]. The theory has been greatly developed in the past three decades, especially in the past ten years, see e.g. [4, 5, 8, 11, 12, 14].

To the best of our knowledge, the study of the dynamics of piecewise smooth maps in the Filippov sense still is not developed. The aim of this paper is to start this study. As we will see along the paper we can have chaotic dynamics even if the piecewise smooth system is the simplest possible: a piecewise affine system.

2 Sliding dynamics

Let $H : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function having $0 = H(0, 0)$ as a regular value. Denote $\Sigma = H^{-1}(0)$, $\Sigma^+ = H^{-1}(0, \infty)$ and $\Sigma^- = H^{-1}(-\infty, 0)$. Before we start the discussion

[1]buzzi@ibilce.unesp.br
[2]medrado@mat.ufg.br
[3]prs@ibilce.unesp.br

about sliding dynamics for piecewise smooth maps we recall the Filippov convention for piecewise smooth vector fields. Consider $X, Y : \mathbb{R}^2 \to \mathbb{R}^2$ smooth vector fields and define the piecewise smooth vector field (PSVF) by

$$Z(x,y) = \begin{cases} X(x,y), & if \quad (x,y) \in \Sigma^+, \\ Y(x,y), & if \quad (x,y) \in \Sigma^-. \end{cases}$$

In what follows we will use the notation $X.H(p) = \langle \nabla H(p), X(p) \rangle$ and $X^i.H(p) = \left\langle \nabla(X^{i-1}.H)(p), X(p) \right\rangle, i \geq 2$, where $\langle .,. \rangle$ is the usual inner product in $\mathbb{R}^2$. Following the Filippov rule, we distinguish the following regions on the discontinuity set Σ :

- *Crossing region for PSVF:* $\Sigma_c = \{p \in \Sigma \,|\, (X.H(p))(Y.H(p)) > 0\}$.
- *Escaping region for PSVF:* $\Sigma_e = \{p \in \Sigma \,|\, X.H(p) > 0$ and $Y.H(p) < 0\}$.
- *Sliding region for PSVF:* $\Sigma_s = \{p \in \Sigma \,|\, X.H(p) < 0$ and $Y.H(p) > 0\}$.

Consider $Z = (X, Y)$ and $p \in \Sigma_e \cup \Sigma_s$. The *sliding vector field* Z^Σ associated to Z at p is the convex combination of $X(p)$ and $Y(p)$ tangent to Σ at p (see Figure 1).

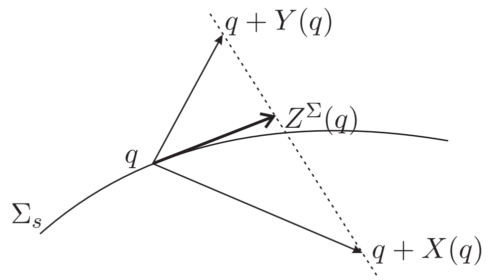

Figure 1: Filippov's convention.

Consider two $F^+, F^- : \mathbb{R}^2 \to \mathbb{R}^2$ smooth maps. The space state $\mathbb{R}^2$ is splited into two subsets Σ^+ and Σ^- by Σ. We refer Σ as *switching manifold* and without loss of generality we assume that $\Sigma = \{(x,y) \in \mathbb{R}^2; x = 0\}$. We denote $F = (F^+, F^-) \in \Omega(\mathbb{R}^2)$, $F^+ = (f_1, g_1)$ and $F^- = (f_2, g_2)$. Define the piecewise smooth map (PSM) by

$$F(x,y) = \begin{cases} F^+(x,y), & if \quad (x,y) \in \Sigma^+, \\ F^-(x,y), & if \quad (x,y) \in \Sigma^-. \end{cases} \tag{1}$$

Inspired in the Filippov rule for vector fields we define the following regions in Σ:

- *Crossing region for PSM:* $\Sigma_c = \{p \in \Sigma \,| \, : f_1(p).f_2(p) > 0\}$.
- *Escaping region for PSM:* $\Sigma_e = \{p \in \Sigma \,|\, f_1(p) > 0$ and $f_2(p) < 0\}$.
- *Sliding region for PSM:* $\Sigma_s = \{p \in \Sigma \,|\, f_1(p) < 0$ and $f_2(p) > 0\}$.

Now we define the sliding map associated to a piecewise smooth map.

Definition 2.1. *Consider $F = (F^+, F^-)$ and $p \in \Sigma_e \cup \Sigma_s$. The sliding map F^Σ associated to F at p is defined by a convex combination $(1-\lambda)F^+ + \lambda F^-, \lambda \in (0,1)$ satisfying that the first coordinate is zero, that is*

$$F^\Sigma(p) = \left(0, \frac{f_2(p)g_1(p) - f_1(p)g_2(p)}{f_2(p) - f_1(p)}\right). \tag{2}$$

3 One dimensional chaotic dynamics

There are many possible definitions of chaos, ranging from measure theoretic notions of randomness in ergodic theory to the topological approach we will adopt here. The first ingredient to a dynamical system to be chaotic is the following definition.

Definition 3.1. *A dynamical system $f : J \to J$ is said to be topologically transitive if for any pair of open sets $U, V \subset J$ there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$. Here f^k means $f \circ f \circ \cdots \circ f$, k times.*

If a map is topologically transitive then it has points that eventually move under iteration from one arbitrarily small neighborhood to any other. In other words, the dynamical system can not be decomposed into two disjoint connected components.

The second ingredient to a dynamical system be chaotic is the following.

Definition 3.2. *A dynamical system $f : J \to J$ has sensitive dependence on initial conditions if there exists $\delta > 0$ such that, for any $x \in J$and any neighborhood V of x, the exists $y \in U$ and $n \geq 0$ such that $|f^n(x) - f^n(y)| > \delta$.*

If a map possesses sensitive dependence on initial conditions then there exist points arbitrarily close to x which eventually separate from x by at least δ under iteration of f. The systems that satisfy this definition are very complicated for practical purposes. Small errors in computation introduced by round-off may become very large and can conduce to wrong conclusions.

We will see that the third ingredient of chaos is a regularity of the system in terms of periodic points.

There are many possible definitions of chaos, some stronger and some weaker than ours. We choose the Devaney's Definition of Chaos that can be seen in [6].

Definition 3.3. *Let V be a set and $f : J \to J$ be a map. The map f is said to be chaotic on J if*

1. *f has sensitive dependence on initial conditions.*
2. *f is topologically transitive.*
3. *periodic points are dense in J.*

We have not mentioned before, but a point $p \in J$ is periodic to the map $f : J \to J$ if there exists $n > 0$ such that $f^n(p) = p$.

A very well studied dynamical system is the quadratic family $F_\mu(x) = \mu x(1 - x)$ that depends on the parameter $\mu > 1$. When the parameter vary from 1 to 3 then F_μ has trivial dynamics. But for $\mu > 2+\sqrt{5}$ the dynamics is very complicated. The proof of next proposition can be found in [6].

Proposition 3.1. *The quadratic maps $F_\mu(x) = \mu x(1 - x)$ are chaotic on Λ when $\mu > 2 + \sqrt{5}$. Here Λ is a Cantor Set contained in $[0, 1] \subset \mathbb{R}$.*

In next section we give more details about the Cantor Set.

4 Chaotic sliding dynamics

In this section we present our main result. In fact is a very surprising result because of the following. The affine maps are known to have trivial dynamics, but in the context of piecewise affine maps, the switching manifold can produce chaotic sliding dynamics, according to Definition 3.3.

Theorem 4.1. *There exists a piecewise affine map $F = (F^+, F^-)$, given by (1), such that the sliding map F^Σ associated to F, given by (2), is chaotic on a subset C of the switching manifold Σ.*

Proof. Consider the piecewise affine map given by

$$F(x, y) = \begin{cases} F^+(x, y) = (-2 + a^-x + y, 2 + d^-x - 9y), & if \quad x > 0, \\ F^-(x, y) = (1 + a^+x + y, -1 + d^+x + 15y), & if \quad x < 0. \end{cases} \tag{3}$$

The sliding region Σ_s is given by conditions $f_1(0, y) < 0$ and $f_2(0, y) > 0$, i.e., $-2 + y < 0$ and $1 + y > 0$. So, we have that

$$\Sigma_s = \{0\} \times (-1, 2).$$

See Figure 2.

Now, we compute the sliding map F^Σ associated to F at $p = (0, y)$ according to (2) and we obtain

$$\begin{aligned} F^\Sigma(p) &= \left(0, \frac{f_2(p)g_1(p) - f_1(p)g_2(p)}{f_2(p) - f_1(p)}\right) \\ &= \left(0, \frac{(1 + y)(2 - 9y) - (-2 + y)(-1 + 15y)}{(1 + y) - (-2 + y)}\right) \\ &= (0, 8y(1 - y)). \end{aligned}$$

Observe that F^Σ has essentially the same expression of the quadratic family F_μ, with $\mu = 8$. So, using the fact that $8 > 2+\sqrt{5}$, it follows from Proposition 3.1 that F^Σ is chaotic on a Cantor set $C = \{0\} \times \Lambda$. We have that $C \subset \{0\} \times [0, 1] \subset \{0\} \times (-1, 2) \subset \Sigma$. $\square$

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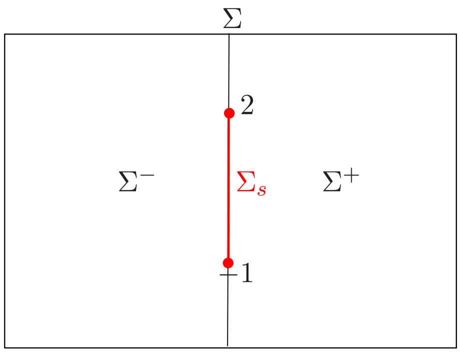

Figure 2: Sliding Region.

Just for sake of completeness, in the sequel we give details of the Cantor set Λ. The interval $I = [0, 1]$ is no longer invariant under F_μ, for $\mu > 2 + \sqrt{5}$, i.e., there are points which are mapped outside I. It is still possible to consider the dynamics of F_μ, but one has to restrict the domain to an invariant subset of I, i.e. to the set Λ of the form

$$\Lambda = \bigcap_{n \in \mathbb{N}} F_\mu^{-n}(I).$$

If $y \in C \subset I$ for each $n \in \mathbb{N}$, we have that $\mathcal{O}(y) = \{y, F_\mu(y), F_\mu^2(y), \dots\} \subset I$. This set is called the invariant set of the map F_μ and it turns out to be non-empty. The map F_μ on the space $X = \Lambda$ gives a well defined dynamical system, since $F_\mu(\Lambda) \subset \Lambda$ and for any $y \in \Lambda$ we can iterate F_μ forever. The set Λ, though, has a quite complicated fractal structure: it is a Cantor set. The best known Cantor set is the middle third Cantor set.

Let us describe the invariant set of F_μ iteratively. One can see that the points y such that $F_\mu(y) \in I$ belong to the two disjoint subintervals, say I_1 and I_2 such that $F_\mu^{-1}([0, 1]) = I_1 \cup I_2$. The points for which $F_\mu(y) \in I$ and $F_\mu^2(y) \in I$ belong to $F_\mu^{-1}(I_1) \cup F_\mu^{-1}(I_2)$ which consists of 4 disjoint intervals, two obtained by removing a central subinterval from I_1 and the other two obtained by removing a central interval from I_2. Continuing like this, one can see that the points which can be iterated n times belong to a disjoint union of 2^n intervals. The set of points which can be iterated infinitely many times can be obtained by iterating this construction. What is left by intersecting all the disjoint unions of 2^n intervals is also a Cantor set and has a fractal structure.

5 Conclusions

The contribution of this work is in the theory of piecewise smooth systems, particularly in the case of piecewise affine maps. More specifically, we have obtained chaotic dynamics for the sliding map associated to a particular piecewise affine map.

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