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Roots of Some Trinomial Equations

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Abstract. The main purpose of this paper is to determine the behavior of the roots of a kind of trinomial equation that appears in certain financial mathematics problems. In addition, we present the regions of the complex plane where these roots are located.
Keywords. Roots, Trinomial equations, Financial mathematics.

1 Introduction

The trinomial equations of degree n, represented by

$$P(z) = z^n + \alpha z^m + \beta = 0, \tag{1}$$

with $m < n$ (m and n natural) and α and β real, was studied by important names of Mathematics, as Lambert [2] and Euler [6], for example. But at the end of the nineteenth century and beginning of twentieth century ocurred some advances in this area, listed in the following [8]: Nekrasov, in 1887, determined sectors in the complex plane, where each sector contained a root of the trinomial equation; Bohl, in 1914, found a method for calculating the number of roots of a trinomial equation in a given circle in the complex plane; Herglotz, in 1922, studied the Riemann surfaces that correspond to the trinomial equations; Egerváry, in 1930, presented a study of results on the arrangements of the roots of trinomial equations. Recent publications show more specific results about the location of the zeros of special classes of lacunary polynomials (see, for example, [3]).

For some values of m, α and β, equation (1) is used in some problems of financial mathematics, related to determine the interest rate of a uniform serie of payment. For example, considering the number of periods n, the payment PMT and the future value FV, the interest rate I of a uniform serie of payment is obtained by the equation

$$PMT = \frac{FV \times I}{(1+I)^n - 1}. \tag{2}$$

More details can be found in [1, 9].

A solution of $n > 4$ can only be obtained by approximation. Actually, computers, using numerical algorithms, can solve this problem very quickly. But this fact has not diminished the algebraic beauty of the problem.

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From $z = 1 + I$, the equation (2) can be rewrite as

$$z^n + \alpha z - (1 + \alpha) = 0, \tag{3}$$

with $\alpha = -\dfrac{FV}{PMT}$. As $FV > PMT$, we have $\alpha < -1$.

In this paper we study the behavior of the roots of equation (3) using classical results of literature and we determine the number of real and non-real roots. Furthermore, the location of these roots is established.

2 Preliminary Results

In this section we present classical results of literature that will be used to show the main results of this paper.

More details of the following result can be found in [7].

Theorem 2.1 (Descartes' Rule of Signs). *Let Z be a number of positive zeros of a polynomial $P(z) = a_0 + a_1 z + \ldots + a_n z^n$ and C the number of changes of sign of the sequence of coefficients. Hence, $C - Z \geq 0$ and $C - Z$ is an even number.*

The following results can be found in [4, 5].

Theorem 2.2 (Eneström-Kakeya). *Let $P(z) = a_0 + a_1 z + \ldots + a_n z^n$ be any polynomial whose coefficients satisfy*

$$a_0 > a_1 > \ldots > a_n > 0.$$

Then $P(z)$ has no zeros for $|z| < 1$.

Theorem 2.3. *Let $P(z) = a_0 + a_1 z + \ldots + a_n z^n$ be a complex polynomial and let r be the unique positive root of equation*

$$f(z) = |a_n| z^n - (|a_{n-1}| z^{n-1} + \ldots + |a_1| z + |a_0|) = 0.$$

Then all the zeros of $P(z)$ lie in the circle $|z| \leq r$.

3 Main Results

Firstly, observe that $z = 1$ is root of the equation (3).

Lemma 3.1. *About the zeros of $P(z) = z^n + \alpha z - (1 + \alpha)$ ($\alpha \in \mathbb{R}$, $\alpha < -1$), we have:*

1. *for n even, $P(z)$ has two positive zeros and $n - 2$ non–real zeros;*
2. *for n odd, $P(z)$ has two positive zeros, one negative zero and $n - 3$ non–real zeros.*

Proof. From Descartes's rule of signs, $P(z) = 0$ has zero or two positive roots. As $P(1) = 0$, we conclude that $P(z)$ has two positive zeros, $z = 1$ and $z = a$. Applying the Descartes's rule of signs to $P(-z)$, follows that $P(z)$ has one negative zero for n odd and $P(z)$ has no negative zero for n even.

Hence, for n even, $P(z)$ has two positive zeros and $n - 2$ non–real zeros and, for n odd, $P(z)$ has two positive zeros, one negative zero and $n-3$ non–real zeros. ☐

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Lemma 3.2. *The equation* $z^n + \alpha z - (1+\alpha) = 0$ *(*$\alpha \in \mathbb{R}$, $\alpha < -1$*) can be represented by*

$$P(z) = z^n + \alpha z - (1+\alpha) = (z-1)Q(z),$$

where

$$Q(z) = z^{n-1} + z^{n-2} + ... + z + (1+\alpha) = (z-a)R(z),$$

with $a \neq 1$ *the positive zero of* $P(z)$ *and*

$$R(z) = z^{n-2} + (a+1)z^{n-3} + (a^2+a+1)z^{n-4} + ... + (a^{n-3} + ... + a + 1)z + (a^{n-2} + ... + a + 1).$$

Proof. Follows directly by simple manipulations. □

Theorem 3.1. *The zeros of* $P(z) = z^n + \alpha z - (1+\alpha)$ *(*$\alpha \in \mathbb{R}$, $\alpha < -1$*) lie in*

$$1 \leq |z| \leq \delta, \tag{4}$$

where δ *is the unique positive zero of*

$$f(z) = z^n - (|\alpha|z + |1+\alpha|).$$

Proof. The inequality $|z| \geq 1$ follows directly from the Eneström-Kakeya Theorem and Lemma 3.2. The inequality on the right side of (4) is obtained from Theorem 2.3. □

In Theorem 3.1 observe that, for n odd, $\delta = |s|$, where s is the unique negative zero of $P(z)$.

4 Numerical Examples

For some values of n, FV and PMT, in Table 1 we can see the coefficients and zeros of $P(z)$ and the interest rate I.

Figs. 1 and 2 display the zeros of $P(z) = \dfrac{553}{47} - \dfrac{600}{47}z + z^8$ and $P(z) = \dfrac{1561}{64} - \dfrac{1625}{64}z + z^9$, respectively, represented by •. We can observe that the zeros satisfy the conditions of the results presented in the previous section, where $\delta = 1.5386$ in the first case and $\delta = 1.5894$ in the second case.

As $z = 1 + I$, considering $z = z_8 = 1.1302$, we have $I = 0.1302$ in the first case and, in the second case, for $z = z_9 = 1.2463$, we have $I = 0.2463$.

5 Conclusion

In this paper we present properties of the roots of a special class of trinomial equations, which is very important in some problems of financial mathematics.

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n	FV	PMT	$P(z)$	Zeros of $P(z)$	I (%)
8	12000,00	940,00	$P(z) = \frac{553}{47} - \frac{600}{47}z + z^8$	$z_1 = -1.397 - 0.634i$, $z_2 = -1.397 + 0.634i$, $z_3 = -0.429 - 1.433i$, $z_4 = -0.429 + 1.433i$, $z_5 = 0.761 - 1.182i$, $z_6 = 0.761 + 1.182i$, $z_7 = 1$ and $z_8 = 1.1302$	13,02
9	6500,00	256,00	$P(z) = \frac{1561}{64} - \frac{1625}{64}z + z^9$	$z_1 = -1.58945$, $z_2 = -1.1532 - 1.07521i$, $z_3 = -1.1532 + 1.07521i$, $z_4 = -0.103491 - 1.53166i$, $z_5 = -0.103491 + 1.53166i$, $z_6 = 0.928277 - 1.11355i$, $z_7 = 0.928277 + 1.11355i$, $z_8 = 1$ and $z_9 = 1.24627$	24,63

Table 1: Numerical examples.

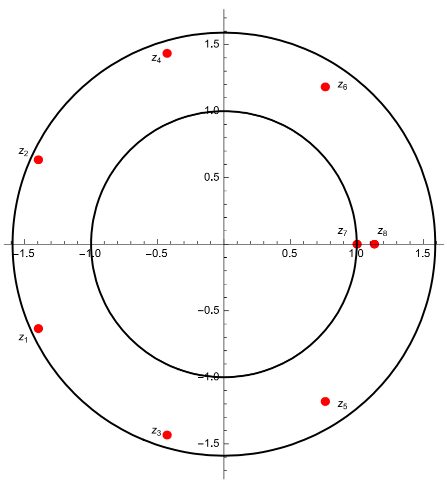

Figure 1: Zeros of $P(z) = \frac{553}{47} - \frac{600}{47}z + z^8$.

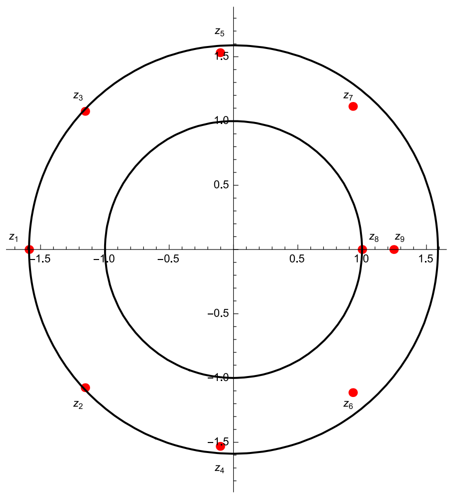

Figure 2: Zeros of $P(z) = \frac{1561}{64} - \frac{1625}{64}z + z^9$.

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