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About Limit Cycles in Continuous Piecewise Linear Differential Systems

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Abstract. In 2012, Lima and Llibre in [3] have studied a class of planar continuous piecewise linear vector fields with three zones. This class can be separated in four other classes and they proved, using the Poincaré map, that this particular class admits always a unique hyperbolic limit cycle. Here, we extended this study for other classes. We proved that some of them also admit always a unique hyperbolic limit cycle, moreover, we find a class that does not have limit cycles and prove the appearance of two limit cycles with one of these cycles appear by perturbations of a period annulus.

Keywords. piecewise linear vector fields, Poincaré map, limit cycles, center, focus.

1 Introduction

We know that, in the qualitative theory of differential systems, the study of limit cycles is one of the most important and studied problem. The maximum number, stability and position of limit cycles are the problems focused, see for instance [8].

In the piecewise continuous context this problem has been studied by many authors and numerous applications can be cited, see for instance [2], [6] and [1].

In this context, specially for piecewise linear differential systems, many works have been developed. Most of them obtaining results on the existence and uniqueness of limit cycles for systems with only one curve of discontinuity. For systems with more then one curve of discontinuity not many works are available and, more important than this, recently (see [7]) an example with more then one limit cycle could be obtained for a special class of Liénard piecewise linear differential system with two curves of discontinuity.

In this paper we improve the results of [3] considering cases not covered and provide a family of piecewise linear differential systems with at least two limit cycles. We observe that the bifurcation that give rise to the second limit cycle is very close to the one that

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appear in [7], namely, one limit cycle visiting the three zones and the second limit cycle visiting two zones and that bifurcates of a period annulus.

2 Definitions and Notations

Consider the plane $\mathbb{R}^2$ divided in three closed regions R_-, R_o and R_+ which frontier are given by two parallel straight lines L_- and L_+ symmetric with respect to the origin such that $(0,0) \in R_o$ and the regions R_- and R_+ have as boundary the respective straight lines L_- and L_+.

Consider the family of differential systems

$$x' = \begin{cases} A_- x + B_- & x \in R_-, \\ A_o x + B_o & x \in R_o, \\ A_+ x + B_+ & x \in R_+, \end{cases} \tag{1}$$

that are continuous piecewise linear differential systems with tree zones where $\mathbf{x} = (x, y) \in \mathbb{R}^2$, $A_i \in \mathcal{M}_2(\mathbb{R})$, $B_i \in \mathbb{R}^2$, $i \in \{-, o, +\}$, and $x' = \dfrac{dx}{dt}$ with t the time.

Let $X_i = A_i x + B_i$, $i \in \{-, o, +\}$, the linear vector fields given in (1). We say that the vector field X_i has a *real equilibrium* x^* in R_i with $i \in \{-, o, +\}$ if x^* is an equilibrium of X_i and $x^* \in R_i$. Otherwise, we will say that X_i has a *virtual equilibrium* x^* in R_i^c if x^* is an equilibrium of X_i and $x^* \in R_i^c$, where R_i^c denotes the complementary of R_i in $\mathbb{R}^2$.

We denote by d_i the determinant of the matrix A_i, by t_i its trace and by $\gamma_i = \dfrac{\alpha_i}{\beta_i}$, for $i \in \{-, o, +\}$ where α_i and β_i are respectively the real and imaginary parts of the eigenvalues of A_i. Furthermore, we assume the following hypothesis:

(H1) X_o has a focus.

(H2) The others equilibria of X_- and X_+ are a center and a focus with different stability with respect to the focus of X_o.

The main results of this paper are the following.

Theorem 2.1. *Assume that system (1) satisfies assumptions (H1), (H2).*

1. *If X_o has a virtual focus and X_+ (respectively X_-) has a real center. Then system (1) has a unique limit cycle, which is hyperbolic.*

2. *If X_o has a real focus at the boundary of R_o. Then system (1) has a unique limit cycle, which is hyperbolic. Except when the focus of X_o belongs to L_+ (respectively L_-) and X_+ (respectively X_-) also has a focus and both foci give rise to a center for system* (1). *In this case system* (1) *has no limit cycles.*

Theorem 2.2. *Assume that system (1) satisfies assumptions (H1), (H2) and X_o has a real focus at the boundary of R_o. If the focus of X_o belongs to L_+ (respectively L_-) and X_+ (respectively X_-) also has a focus at the same point of L_+ (respectively L_-) and both*

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the foci give rise to a center for system (1)*, then at least two limit cycles can appear by small perturbations of the parameters of system* (1)*.*

3 Normal Form and Poincaré Map

The following result, proved in [3], give us a convenient normal form to write system (1) with the number of parameters reduced.

Lemma 3.1. *Suppose that system (1) is such that* $d_o > 0$. *Then there exists a linear change of coordinates that writes system (1) into the form* $\dot{x} = X(x)$, *with* $L_- = \{x = -1\}$, $L_+ = \{x = 1\}$, $R_- = \{(x, y) \in \mathbb{R}^2;\ x \leq -1\}$, $R_o = \{(x, y) \in \mathbb{R}^2;\ -1 \leq x \leq 1\}$, $R_+ = \{(x, y) \in \mathbb{R}^2;\ x \geq 1\}$ *and*

$$X(x) = \begin{cases} A_-x + B_- & x \in R_-, \\ A_ox + B_o & x \in R_o, \\ A_+x + B_+ & x \in R_+, \end{cases} \tag{2}$$

where $A_- = \begin{pmatrix} a_{11} & -1 \\ 1 - b_2 + d_2 & a_1 \end{pmatrix}$, $B_- = \begin{pmatrix} a_{11} \\ d_2 \end{pmatrix}$, $A_o = \begin{pmatrix} 0 & -1 \\ 1 & a_1 \end{pmatrix}$, $B_o = \begin{pmatrix} 0 \\ b_2 \end{pmatrix}$, $A_+ = \begin{pmatrix} c_{11} & -1 \\ 1 + b_2 - f_2 & a_1 \end{pmatrix}$ *and* $B_+ = \begin{pmatrix} -c_{11} \\ f_2 \end{pmatrix}$. *The dot denotes derivative with respect to a new time* s.

We will rewrite the problem of finding limit cycles that visit the three zones R_i, $i \in \{-, o, +\}$ or even only two of them in terms of finding the fixed points of an appropriated Poincaré return map. These Poincaré maps will be defined in the transversal sections $L^O_\pm = \{(\pm 1, y);\ y \geq 0\}$ and $L^I_\pm = \{(\pm 1, y);\ y \leq 0\}$, and we will do a convenient parametrization in the transversal sections $L^O_\pm$ and $L^I_\pm$ assuming $b_2 \neq \pm 1$.

We parametrize L^O_- by the parameter c defined as follows. Let $p_- = (-1, 0)$ be the contact point of X_- with L_- and $\dot{p}_- = X_-(p_-) = (0, b_2 - 1)$. Given, $p \in L^O_-$ we take $c \geq 0$ as the unique non-negative real satisfying $p = p_- - c\dot{p}_-$.

Analogously, we parametrize L^I_- by the parameter d, L^O_+ by the parameter b and L^I_+ by the parameter a.

For study the limit cycles of system (2) that visit the three zones R_i, $i \in \{-, o, +\}$, the Poincaré return map Π is defined on L^O_-, this map involves all the vector fields X_i, $i \in \{-, o, +\}$, and has the form

$$\Pi = \bar{\pi}_o \circ \pi_+ \circ \pi_o \circ \pi_-,$$

where the Poincaré maps $\bar{\pi}_o$, π_+, π_o, and π_- are defined by the respective flows and $\bar{\pi}_o : L^O_+ \to L^O_-$, $\pi_+ : L^I_+ \to L^O_+$, $\pi_o : L^I_- \to L^I_+$, and $\pi_- : L^O_- \to L^I_-$.

Now the study of Π, corresponds to study qualitative behavior of each one of these maps $\bar{\pi}_o$, π_+, π_o, and π_-.

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4 Proof of Theorem 2.1

Part 1.

Following the notations of previous section, we have that $\gamma_+ = 0$ and $\gamma_-, \gamma_o \neq 0$, then we suppose $\gamma_o > 0$ and $\gamma_- < 0$. The case $\gamma_o < 0$ and $\gamma_- > 0$ is analogous.

By Propositions 5 and 6 of [4], the first return map $\Pi : [0, +\infty) \to [c^*, +\infty)$ is well defined, where $c^* = \Pi(0) > 0$ and by Propositions 2, 4 and 6 of [4], π_+ is the identity map and we have

$$\Pi'(c) = \bar{\pi}_o'(\pi_o(\pi_-(c))) \cdot \pi_o'(\pi_-(c)) \cdot \pi_-'(c) = \frac{c}{\Pi(c)} e^{2[\gamma_o(\bar{\tau}_o + \tau_o) + \gamma_- \tau_-]}.$$

with $\tau_- \in (0, \pi)$ increasing with c, and $\bar{\tau}_o + \tau_o \in (0, \pi)$ decreasing with c. So $\gamma_o(\bar{\tau}_o + \tau_o) + \gamma_-\tau_-$ is a decreasing function in c and $\gamma_o(\bar{\tau}_o + \tau_o) + \gamma_-\tau_- \to \gamma_-\pi < 0$ when $c \to \infty$.

Consider the function $h(c) = \Pi(c) - c$. By Propositions 4,5 and 6 of [4],

$$\lim_{c \to \infty} \Pi'(c) = e^{\gamma_- \pi}.$$

Hence $\lim_{c \to \infty} h'(c) = e^{\gamma_- \pi} - 1 < 0$ and by Mean Value Theorem $\lim_{c \to \infty} h(c) = -\infty$. Therefore as $h(0) > 0$ it follows that h has a zero, i.e. Π has a fixed point c_s. Then

$$\Pi'(c_s) = e^{2[\gamma_o(\bar{\tau}_o + \tau_o) + \gamma_- \tau_-]}. \tag{3}$$

As $\Pi(0) > 0$ we have that $\Pi'(c_s) \leq 1$, i.e., c_s is hyperbolic.

Now if we suppose there exist another fixed point c_r. As $\Pi'(c_s) \leq 1$, using the monotonicity of the function $\gamma_o(\bar{\tau}_o + \tau_o) + \gamma_-\tau_-$, the possibilities of signals to the functions $\gamma_o(\bar{\tau}_{os} + \tau_{os}) + \gamma_-\tau_{-s}$ and $\gamma_o(\bar{\tau}_{or} + \tau_{or}) + \gamma_-\tau_{-r}$ and Mean Value Theorem, we guarantee this is not possible.

Part 2

We assume $b_2 = -1$ and distinguish two cases, when X_+ has a center and X_- a focus or when X_+ has a focus and X_- a center. In the first case, the proof is analogous to proof of part 1.

In the second case, we denote by $a_+^* = (\pi_+)^{-1}(b_o^*)$ and $\tilde{\pi}_o(b_o^*) = a_o^*$. We distinguish three cases $a_+^* < a_o^*$, $a_+^* > a_o^*$ and $a_+^* = a_o^*$. Firstly, in all cases it is not possible to have a limit cycle that visit only the regions R_o and R_+ because a closed orbit that visit these two regions correspond to fixed points of the map $\Pi : (0, a_+^*] \to (0, a_+^*]$ given by $\Pi(b) = \tilde{\pi}_o \circ \pi_+(b) = be^{(\gamma_o + \gamma_+)\pi}$, i. e., $\gamma_o = -\gamma_+$ and $\Pi = \tilde{\pi} \circ \pi_+$ is the identity map. Therefore, in the cases $a_+^* < a_o^*$ and $a_+^* > a_o^*$ we have no limit cycles visiting only the regions R_o and R_+ and in the case $a_+^* = a_o^*$ we have a continuous of closed orbits.

Now study the existence of limit cycles that visit the three zones in the cases $a_+^* < a_o^*$ and $a_+^* > a_o^*$ are analogous to proof of part 1.

For the case $a_+^* = a_o^*$, we have, $\Pi'(c) = \dfrac{c}{\Pi(c)} e^{2\gamma_+[\pi - (\bar{\tau}_o + \tau_o)]}$, with $\bar{\tau}_o + \tau_o \in (0, \pi]$ decreasing with c. So $\gamma_+[\pi - (\bar{\tau}_o + \tau_o)]$ is a decreasing function in c and $\gamma_+[\pi - (\bar{\tau}_o + \tau_o)] \to \gamma_+\pi < 0$ when $c \to \infty$.

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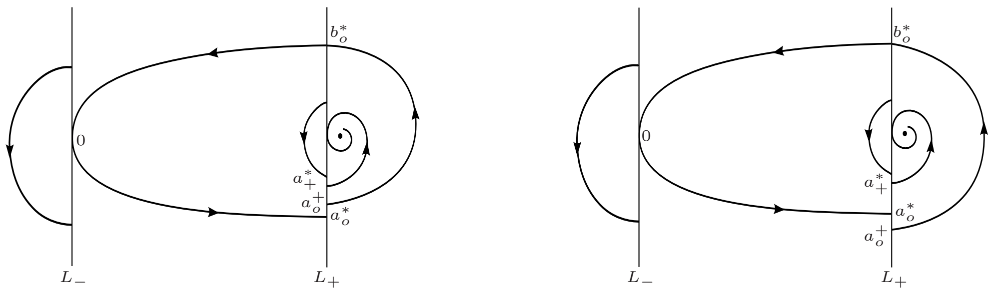
Now the derivatives of Π at 0 give us $\Pi'(0) = 1$ and $\Pi''(0) = \dfrac{4\gamma_+}{3\beta_o} < 0$. If c_s be the smallest nonzero fixed point of Π, as $\Pi'(0) = 1$, we have $\Pi(c) < c$ for all $c \in (0, c_s)$. Therefore $\gamma_+[\pi - (\bar{\tau}_{os} + \tau_{os})] \geq 0$, but this is not possible, because $\gamma_+[\pi - (\bar{\tau}_o + \tau_o)]$ is decreasing and null in zero, i.e. there not exist nonzero fixed point for Π in this case.

5 Proof of Theorem 2.2

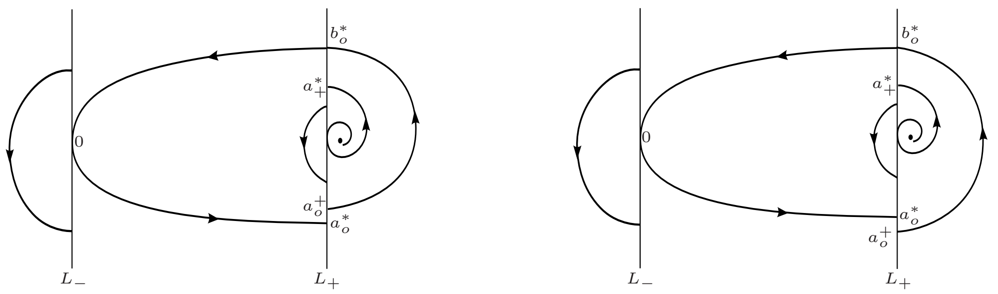
We will assume that $b_2 \leq -1$ and X_+ has a real focus, then X_- has a virtual center and X_o has either a virtual focus (when $b_2 < -1$) or a real focus at $(1, 0)$ (when $b_2 = -1$), thus $\gamma_- = 0$ and $\gamma_+\gamma_o < 0$.

Denote by $a_o^* = \pi_o(0)$, $b_o^* = \bar{\pi}_o^{-1}(0)$ and $a_+^* = \pi_+(0)$. Hence, the orbits of the periodic annuls are broken and give us the four possible phase portraits described in the Figures 1 and 2, with $a_o^+ = \pi_+^{-1}(b_o^*)$.



Case (a) Case (b)

Figure 1: Phase portraits when the periodic annulus is broken and $\gamma_+ < 0$.



Case (a) Case (b)

Figure 2: Phase portraits when the periodic annulus is broken and $\gamma_+ > 0$.

We will consider the Poincaré maps defined in two and three zones respectively and study the sign of the displacement function

$$a_o^* - a_o^+.$$

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If $\gamma_o > 0$, $\gamma_+^* < 0$ and $\gamma_o + \gamma_+^* > 0$, then $a_o^* - a_o^+ > 0$, so we have the case showed in Figure 1 (a). The orbit of system (2) by the point associated to a_o^+, spirals toward the focus of X_+ when $t \to -\infty$. But the focus of X_+ is an attractor, then there is at least a limit cycle in two zones that pass by a point of L_+^I between a_+^* and a_o^+. Moreover this limit cycle is repeller.

In the three zones of Figure 1 (a), we consider the Poincaré map $\Pi : [0, \infty) \to [0, \infty)$ given by $\Pi = \bar{\pi}_o \circ \pi_+ \circ \pi_o \circ \pi_-$. We have that π_- is identity and

$$\lim_{c \to +\infty} \Pi'(c) = e^{\gamma_+ \pi}. \tag{4}$$

As $\gamma_+^* < 0$, so $\gamma_+ < 0$ for $|b_2 + 1|$ small enough and Π is decreasing in a neighborhood of infinity, i.e. the infinity is a repeller to system (2). On the other hand, the orbit $\Gamma(t)$ spirals moving away from the focus. Therefore we have at least a limit cycle in the three zones.

Now if $\gamma_o < 0$, $\gamma_+^* > 0$ and $\gamma_o + \gamma_+^* < 0$, we have $a_o^* - a_o^+ < 0$, i.e. we have the case showed in Figure 2 (b). Therefore, similar the previous case, there is at least a limit cycle in two zones that pass by a point of L_+^O between a_+^* and b_o^*, which is attractor.

Now, for Figure 2 (b), for equation 4 and using that $\gamma_+^* > 0$, we have that the infinity is an attractor to system (2) and as in the previous case we have at least a limit cycle in the three zones.

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