A supnorm estimate for one-dimensional porous medium equations with advection

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\textbf{Abstract.} We give a short derivation of supnorm estimates for solutions of one-dimensional porous medium equations of the form
\[ u_t + (f(t,u))_x = (|u|^\alpha u_x)_x, \]
assuming initial data \( u(\cdot, 0) \in L^{p_0}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) for some \( 1 \leq p_0 < \infty \).

\textbf{Key-words.} Porous Medium Equation, Supnorm Estimate, Comparison Theorem

\section{Introduction}

There are a number of physical applications where the porous medium equation describes processes involving fluid flow, heat transfer or diffusion [5]. The porous medium equation without advection is given by
\[ \frac{\partial u}{\partial t} = (|u|^{m-1}u)_x + f, \quad m > 1, \]  
where \( f = f(x,t) \) is a source term.

Here we consider the following initial-value problem
\[ \begin{cases} 
    u_t + (f(t,u))_x = (|u|^\alpha u_x)_x, & x \in \mathbb{R}, \quad t > 0, \\
    u(\cdot, 0) = u_0 \in L^{p_0}(\mathbb{R}) \cap L^\infty(\mathbb{R}), & 1 \leq p_0 < \infty,
\end{cases} \]  

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where $\alpha \geq 0$ and $f \in C^1([0, \infty) \times \mathbb{R})$ are given. The solutions of (2) are known to be defined for all $t > 0$ and decay as $t \to \infty$ in several spaces. In this work, we derive a supnorm estimate for the solutions of (2) when considering $u(\cdot, 0)$ in $L^p(\mathbb{R})$, $p = p_0 + \alpha/2$. By solution in some interval $[0, T^*)$, $0 < T^* \leq \infty$, we mean a measurable function $u : \mathbb{R} \times [0, T^*) \to \mathbb{R}$ which is bounded in each strip $\mathbb{R} \times [0, T]$, $0 < T < T^*$, and which solves the equation (2) in distributional sense.

2 Preliminary

An important result to obtain a supnorm estimate also for negative solutions is the following Theorem.

**Theorem 2.1. (Theorem of comparison:)**

Let $u(\cdot, t)$, $v(\cdot, t)$ solutions of the equation (1), with initial value $u_0$, $v_0 \in L^\infty(\mathbb{R})$, respectively, both defined for $0 < t < T$ and limited in the strip $\mathbb{R} \times [0, T]$. Also, if

$$|f(x, t, u) - f(x, t, v)| \leq K_f(M, T)|u - v|, \quad \forall x \in \mathbb{R}, \forall t, 0 \leq t \leq T,$$

then

$$u_0(x) \leq v_0(x) \text{ a.e. } x \in \mathbb{R} \Rightarrow u(x, t) \leq v(x, t) \forall x \in \mathbb{R},$$

for all $t$, $0 < t \leq T$.

The proof of this Theorem is in [2].

2.1 Some importants inequalities

The following inequalities will be important throughout this work.

- For any $p, q$ and $r$ such that $0 < p \leq r \leq \infty$, $1 \leq q \leq \infty$:

  $$\|w\|_{L^r(\mathbb{R})} \leq \tilde{K}(r, q, p)\|w\|^{1-\tilde{\theta}}_{L^p(\mathbb{R})}\|w_x\|_{L^q(\mathbb{R})}^{\tilde{\theta}} \forall w \in C^1_0(\mathbb{R}),$$

  where $\tilde{\theta} = \frac{1-p/r}{1+p(1-1/q)}$, $\tilde{K}(r, q, p) = (2\theta)^{-\tilde{\theta}}$ and $\theta = \frac{1}{1+p(1-1/q)}$.

- $\forall \beta_0 > 0$:

  $$\|u\|_{L^\infty(\mathbb{R})} \leq \left(\frac{2 + \beta_0}{4}\right)\|u\|_{L^{\beta_0}(\mathbb{R})}^{1-\theta}\|u_x\|_{L^2(\mathbb{R})}^{\theta},$$

  where $\theta = \frac{1}{1 + \frac{\beta_0}{4}}$.
2.2 Basic Result

Theorem 2.2. If $u(\cdot,t) \in L^\infty_{\text{loc}}([0,T^*), L^\infty(\mathbb{R}))$ solves problem (2) then

1) $u(\cdot,t) \in L^{p_0}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \ \forall \ t$, $0 < t < T^*$

2) $\|u(\cdot,t)\|_{L^q(\mathbb{R})} \leq \|u_0\|_{L^q(\mathbb{R})} \ \forall \ t$, $0 < t < T^*$ ($\forall \ q, \ p_0 \leq q \leq \infty$)

3) $\|u(\cdot,t)\|_{L^q(\mathbb{R})} \leq \|u(\cdot,t_0)\|_{L^q(\mathbb{R})} \ \forall \ t_0 < t < T^*$, $\forall \ q, \ p_0 \leq q \leq \infty$.

Proof of (1). For simplicity, we will consider the case of positive solutions, which are known to be smooth. Let $\zeta \in C^2(\mathbb{R})$ be such that $\zeta(x) = 1 \ \forall \ |x| \leq 1$, $\zeta(x) = 0 \ \forall \ |x| \geq 2$, $0 \leq \zeta(x) \leq 1 \ \forall \ x \in \mathbb{R}$. Given $R > 0$, let $\zeta_R$ be the cut-off function given by $\zeta_R(x) = \zeta\left(\frac{x}{R}\right)$.

Let $p_0 \leq q < \infty$. Multipliyng the PDE at the initial value problem (2) by $q u^{q-1} \zeta_R(x)$ we have

$$\frac{\partial}{\partial t} u^q \zeta_R(x) + f(t,u)x u^{q-1} \zeta_R(x) = q u^{q-1}(u^\alpha u_x) \zeta_R(x).$$

Integrating on $\mathbb{R} \times [0,t]$, we get

$$\int_{|x|<2R} u(x,t) \zeta_R(x) dx + q(q-1) \int_0^t \int_{|x|<2R} u^{q+\alpha-2} u_x^2 \zeta_R(x) dx d\tau = \int_{|x|<2R} u_0(x)^q \zeta_R(x) dx + \frac{q}{q+\alpha} \int_0^t \int_{|x|<2R} u^{q+\alpha} \zeta_R''(x) dx d\tau - \int_0^t \int_{|x|<2R} f(t,u) x u^{q-1} \zeta_R(x) dx d\tau.$$

Next, integrating by parts and then, letting $R \to \infty$, we get the result.

Proof of (2) and (3). Again, we consider the simpler case of positive solutions. Defining $F(t,U) = \int_0^U f(t,v) v^{q-1} dv$, then equation (4) can be written as

$$\int_{|x|<2R} u(x,t) \zeta_R(x) dx + q(q-1) \int_0^t \int_{|x|<2R} u^{q+\alpha-2} u_x^2 \zeta_R(x) dx d\tau = \int_{|x|<2R} u_0(x)^q \zeta_R(x) dx + \frac{q}{q+\alpha} \int_0^t \int_{|x|<2R} u^{q+\alpha} \zeta_R''(x) dx d\tau + q \int_0^t \int_{|x|<2R} F(t,u) \zeta_R'(x) dx d\tau.$$

Observe that

$$\int_0^t \int_{R<|x|<2R} F(t,u) \zeta_R'(x) dx d\tau \leq \int_0^t \int_{R<|x|<2R} |F(t,u)| |\zeta_R'(x)| dx d\tau \leq \frac{M}{R} \int_0^t \int_{R<|x|<2R} |u(x,\tau)|^q dx d\tau \to 0,$$

when $R \to \infty$, where $M$ is a constant. Then

$$\int_R u(x,t) dx \leq \int_R u(x,t)^q dx + q(q-1) \int_0^t \int_R u(x,\tau)^{q+\alpha-2} u_x^2 dx d\tau \leq \int_R u_0(x)^q dx.$$
Therefore, we get
\[ \|u(\cdot,t)\|_{L^q(\mathbb{R})} \leq \|u_0\|_{L^q(\mathbb{R})} \quad \forall q, \quad p_0 \leq q < \infty, \quad \forall t, \quad 0 < t < T^*, \]
and
\[ \|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})} \quad \forall t, \quad 0 < t < T^*. \]
as claimed. In particular, solutions of the initial-value problem (2) are globally defined (i.e., \(T^* = \infty\)).

3 Main Theorems

**Theorem 3.1.** If \(u(\cdot,t) \in L^\infty_{loc}([0,\infty), L^\infty(\mathbb{R}))\) solves problem (2) with \(u_0 > 0\), then
\[ \|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq K(\alpha, p_0)\|u(\cdot,t_0)\|_{L^p_0(\mathbb{R})}^{2p_0/(\alpha+q)}(t - t_0)^{-\frac{1}{\alpha+q}}, \quad \forall t, \quad 0 \leq t_0 < t, \]
where \(K(\alpha, p_0)\) is a constant that only depends on \(\alpha\) and \(p_0\).

**Proof.** Let \(\psi \in C^1(\mathbb{R})\) be monotonically increasing such that \(\psi(u) = 1 \quad \forall u \geq 1, \quad \psi(0) = 0\) and \(\psi(u) = -1, \quad \forall u \leq -1\). Taking \(\delta > 0\), let us define \(\psi_\delta(u) = \psi(\frac{u}{\delta})\) and \(\phi_\delta(u) = L_\delta(u)^q, \quad q \geq 2\), where \(L_\delta(u) = \int_0^u \psi_\delta(v)dv, \quad L_\delta \in C^2(\mathbb{R})\). Let \(\gamma > 0\). Multiplying the equation in (2) above by \((t - t_0)^\gamma \phi_\delta(u)\) and integrating in \(\mathbb{R} \times [t_0, t]\), we get
\[ \int_{t_0}^t \int_{\mathbb{R}} (\tau - t_0)^\gamma \phi_\delta'(u(x, \tau))(u(x, \tau), v(x, \tau))dx d\tau + \int_{t_0}^t \int_{\mathbb{R}} (t - t_0)^\gamma \phi_\delta(u(x, \tau))(f(\tau, u))x dx d\tau \]
\[ = \int_{t_0}^t \int_{\mathbb{R}} (\tau - t_0)^\gamma \phi_\delta(u(x, \tau))(|u|^\alpha u_x) dx d\tau \]
By Fubini’s theorem, integrating by parts, using an appropriate cut-off function and taking \(\delta \to 0\), this gives
\[ (t - t_0)^\gamma \|u(x, t)\|_{L^q(\mathbb{R})}^q + q(q - 1)\int_{t_0}^t (\tau - t_0)^\gamma \int_{\mathbb{R}} |u(x, \tau)|^{\alpha + q - 2}(u_x)^2 dx d\tau \]
\[ \leq \gamma \int_{t_0}^t (\tau - t_0)^{-1}\|u(x, \tau)\|_{L^q(\mathbb{R})}^q d\tau \]
Introducing
\[ v^q(x, t) := \begin{cases} u(x, t) & \text{se } \sigma = \alpha + q = 2, \\ |u(x, t)|^{\sigma/2} & \sigma = \alpha + q > 2, \end{cases} \]
we then have
\[ (t - t_0)^\gamma \|v^q(\cdot, t)\|_{L^{2q/\sigma}(\mathbb{R})}^{2q/\sigma} + \frac{4q(q - 1)}{(\alpha + q)^2} \int_{t_0}^t (\tau - t_0)^\gamma \|v^q(\cdot, \tau)\|_{L^{2q/\sigma}(\mathbb{R})}^{2q/\sigma} d\tau \]
\[ \leq \gamma \int_{t_0}^t (\tau - t_0)^{-1}\|v^q(\cdot, \tau)\|_{L^{2q/\sigma}(\mathbb{R})}^{2q/\sigma} d\tau \]
Using Hölder, Nirenberg-Sobolev-Gagliardo II (5), with \( \beta_0 = 2q/\sigma \) and \( q = 2p_0 \), and Nirenberg-Sobolev-Gagliardo I (4) inequalities, we obtain the supnorm estimate (6).

Let \( w(\cdot,t) \) be the solution of (2) with initial condition \( w_0 = w_0^+ + \epsilon \zeta \) for some \( \epsilon > 0 \), where \( w_0^+ \) denotes the positive part of \( w_0 \) and \( \zeta \in C^0(\mathbb{R}) \cap L^{p_0}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \). That is, \( w_0 \geq u_0 \). Then, by the Theorem of Comparison (2.1), \( u(\cdot,t) \leq w(\cdot,t) \), for all \( 0 \leq t < T \) and

\[
\|w(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq K(\alpha,p_0)\|w(\cdot,t_0)\|_{L^{p_0}(\mathbb{R})(t-t_0)}^{\frac{2p_0}{\alpha+2p_0}} \|L^{p_0}(\mathbb{R})(t-t_0)^{-\frac{1}{\alpha+2p_0}}, \forall t, 0 \leq t_0 < t, \tag{7}
\]

Now let \( z(\cdot,t) \) be the solution of (2) with initial condition \( z_0 = -u_0^- - \epsilon \zeta \) for some \( \epsilon > 0 \), where \( u_0^- \) denotes the negative part of \( u_0 \). That is, \( z_0 \leq u_0 \). Then, by the Theorem of Comparison (2.1), \( u(\cdot,t) \geq w(\cdot,t) \), for all \( t, 0 \leq t < T \) and

\[
\|z(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq K(\alpha,p_0)\|z(\cdot,t_0)\|_{L^{p_0}(\mathbb{R})(t-t_0)}^{\frac{2p_0}{\alpha+2p_0}} \|L^{p_0}(\mathbb{R})(t-t_0)^{-\frac{1}{\alpha+2p_0}}, \forall t, 0 \leq t_0 < t, \tag{8}
\]

By (7) and (8), we have

\[
\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq K(\alpha,p_0)\max\{\|u_0^+\|, \|u_0^-\|\}^{\frac{2p_0}{\alpha+2p_0}} \|L^{p_0}(\mathbb{R})(t-t_0)^{-\frac{1}{\alpha+2p_0}}, \forall t, 0 \leq t_0 < t.
\]

This proves the following theorem:

**Theorem 3.2.** If \( u(\cdot,t) \in L^{\infty}_{\text{loc}}(\mathbb{R}, L^\infty(\mathbb{R})) \) solves problem (2), then

\[
\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq K(\alpha,p_0)\max\{\|u_0^+\|, \|u_0^-\|\}^{\frac{2p_0}{\alpha+2p_0}} \|L^{p_0}(\mathbb{R})(t-t_0)^{-\frac{1}{\alpha+2p_0}}, \forall t, 0 \leq t_0 < t,
\]

where \( K(\alpha,p_0) \) is a constant that only depends on \( \alpha \) and \( p_0 \) and \( u_0^+ \) and \( u_0^- \) denote the positive and negative part of \( u_0 \), respectively.

### 4 Conclusions

We derived a supnorm estimate for the solution of the porous medium equation (2) with no restriction on the sign of \( u_0 \).

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References


