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Graphs: A global invariant of the stable maps

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In Singularity theory, maps from closed surfaces to the 2−sphere have been much studied (see [1], [6], [7]). An important problem is to search of invariants for the classification of the stable maps. In [6], Ohmoto and Aicardi determined local invariants of stable maps between surfaces, from codimension one transition over the image of the map. Noting the existence of maps with the same apparent contour and different singular sets (from the topological view point over the domain), Hacon-Mendes-Romero introduced in [2], the *dual graph* of stable maps as a global topological invariant of stable maps from a closed orientable surface to the plane. It is associated to the singular set, classifying by full the domain of map. Besides being a topological invariant, the dual graph is a useful tool in the construction of examples of these maps with a predetermined singular set.

Given a stable map f of a closed orientable surface M to the 2−sphere, we denoted by Σf the singular set of f. We can associate to the pair $(M, \Sigma f)$, a graph with weights on the vertices as follows (see [4], [5]): every *regular region* U from $M \setminus \Sigma f$, corresponds to a *vertex* v of the graph, every *curve* α from Σf corresponds to an *edge* a of the graph, a vertex v gets the *weight* w if the corresponding regular region to v has *genus* w (sum of w toros). An edge a connect the vertex v if, and only if, the corresponding singular curve to a lies in the boundary of the regular region corresponding to v.

The number of connected components of Σf is C. We consider by M^+ the union of all the regular region, which f preserves the orientation, including their boundaries, and by M^- the union of all the regular region, which f reverses the orientation, including their boundaries. Let us denote by V^+ (V^-) the number of connected components of M^+ (M^-) and $V = V^+ + V^-$. Finally, W^+ and W^- are the total genus of M^+ and M^-, respectively, then $W = W^+ + W^-$. Clearly, M^+ and M^- meet in their common boundary any singular curve or component of Σf.

In [3], the authors showed that all graph associated to the stable map from the closed orientable surface in the plane (or in the 2−sphere) is a *bipartite* graph, i.e, it is possible attach labels $\pm$ to its vertices in such a way that the vertices at the end of each edge have opposite labels. Observe that if M is an orientable surface, a singular curve of Σf separates two regular regions of $M \setminus \Sigma f$ that are immersed by f on S^2 with opposite orientations. Then a graph with weights in the vertices is a graph of a stable map from closed orientable surface into the 2−sphere if, and only if, the graph is bipartite. In [4], these authors showed that all bipartite graph $\mathcal{G}(V, E, W)$, where V, E, W, corresponds respectively the number of vertices, the number of the edges and total weight in the vertices, can be realized by a stable map from a closed orientable surface M to the 2−sphere, with genus $g(M) = 1 - V + E + W$. In the case of the stable map without points of cusps, known as *fold map*, the authors showed that any non bipartite graph is realized by any fold map from a closed orientable surface M in the 2−sphere with degree $d = (V^+ - V^-) - (W^+ - W^-)$.

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At this work, we extend the results in [4] for any closed surfaces. Note que a closed curve over a non orientable surface may have as neighborhood either a band of Moebius or a cylinder. Given a surface with a set of closed, simple (without intersection itself) and disjoint curves, we denote by S the number of curves which have as neighborhood a band of Moebius. An edge of the graph is a *loop* if the neighborhood of curve is contained in a regular connected component. The loop gets a $\star$ if these neighborhood is the band of Moebius. So, we denote a new graph by $\mathcal{G}^S(V, E, W)$, to indicate that the graph has V vertices, E edges, S loops with $\star$ and total weight in the vertices equal to W. Naturally, all bipartite graph has $S = 0$.

For non-bipartite graphs, we obtain the follows results.

Theorem 0.1. *Any graph $\mathcal{G}^S(V, E, W)$ with weight in the vertices is the graph of a stable map from a closed surface M in the $2-$sphere, where the genus of M is given by $g(M) = 1-V+E+W$ if M is orientable and by $g(M) = 2(1 - V + E + W) - S$ otherwise.*

Theorem 0.2. *Any graph $\mathcal{G}^0(V, E, W)$ is realized by some fold map of a closed surface M to the sphere, where the genus of M is given by $g(M) = 1 - V + E + W$ if M is orientable and by $g(M) = 2(1 - V + E + W)$ if M is non orientable.*

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