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Generalized Atanassov's Intuitionistic Fuzzy Index and Conjugate with S-implications

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Abstract. This work extends the study of properties related to the Generalized Atanassov's Intuitionistic Fuzzy Index, by considering the concept of conjugate fuzzy implications, mainly interested in the class of S-implications.

Keywords. Intuitionistic Fuzzy Index, Intuitionistic Fuzzy Automorphims, S-implications

In order to deal with the available information in fuzzy reasoning systems, the Atanassov's intuitionistic fuzzy approach allows two non-complementary freedom degrees named as membership and non-membership degrees. The flexible relationship between these non complementary membership functions is formally expressed as the Atanassov-intuitionistic fuzzy index (A-IFIx), also called as hesitancy (indeterminance) degree of an element in an Atanassov-intuitionitic fuzzy set. Since there are applications in which experts do not have precise knowledge, it formalizes the expression related to the expert uncertainties or lack of information in identifying a particular membership function. In addition, the A-IFIx provides a measure of the lack of information for or against a given proposition based on Atanassov-intuitionistic fuzzy logic (A-IFL).

Despite so many applications of A-IFI in modelling inference rules in fuzzy reasoning, in [4] a new concept – the Generalized Atanassov's Intuitionistic Fuzzy Index (A-GIFIx) is characterized in terms of fuzzy implication operators which is described by a constructive method making use of automorphisms. In [5], by means of special aggregation functions applied to the A-GIFIx, the Atanassov's intuitionistic fuzzy entropy is discussed and some examples are analysed.

Extending these previous researches, this work contributes with the study of properties related to A-GIFIx, considering the concept of conjugate fuzzy implications, mainly interested in the class of S-implications and corresponding dual constructions. Additionally,

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A-GIFIx associated with both standard negation and well known fuzzy implications are considered: Lukaziewicz, Reichenbach, Gaines-Rescher and I_{30} [9].

The preliminaries describe the basic properties of fuzzy connectives and basic concepts of A-IFL. The study of the A-GIFIx and general results in the analysis of its properties are stated in Section 2. Final remarks are reported in the conclusion.

1 Preliminaries

We firstly give a brief account on FL, keeping this paper self-contained by reporting basic concepts of automorphisms, fuzzy negations on $U = [0,1]$ and main properties of fuzzy implications.

1.1 Fuzzy connectives

By [8, Def. 4.1], an **automorphism** $\phi : U \to U$ is a bijective, strictly increasing function:

A1 : $x \leq y$ iff $\phi(x) \leq \phi(y)$, $\forall x, y \in U$.

In [6], an automorphism $\phi : U \to U$ is a continuous, strictly increasing function such that

A2 : $\phi(0) = 0$ and $\phi(1) = 1$.

Let $Aut(U)$ be the set of all automorphisms. Automorphisms are closed under composition, $\phi \circ \phi' \in Aut(U)$, $\forall \phi, \phi' \in Aut(U)$, and there exists the inverse $\phi^{-1} \in U$, such that

A3 : $\phi \circ \phi^{-1} = id_U$, $\forall \phi \in Aut(U)$.

Thus, $(Aut(U), \circ)$ is a group, with the identity function being the neutral element. The action of an automorphism $\phi : U \to U$ on a function $f : U^n \to U$, called **conjugate of** f, and given by

$$f^{\phi}(x_1, \ldots, x_n) = \phi^{-1}(f(\phi(x_1), \ldots, \phi(x_n))). \tag{1}$$

A function $N : U \to U$ is a *fuzzy negation* (FN) if

N1 : $N(0) = 1$ and $N(1) = 0$; **N2** : If $x \geq y$ then $N(x) \leq N(y)$, $\forall x, y \in U$.

FNs satisfying the involutive property given below are called *strong* fuzzy negations [6]:

N3 : $N(N(x)) = x$, $\forall x \in U$.

By [7], a *fuzzy implication* $I : U^2 \longrightarrow U$ satisfies the conditions:

I1: If $x \leq z$ then $I(x, y) \geq I(z, y)$;
I2: If $y \leq z$ then $I(x, y) \leq I(x, z)$
I3: $I(0, y) = 1$;
I4: $I(x, 1) = 1$
I5: $I(1, 0) = 0$.

Several reasonable properties may be required for fuzzy implications:

I6: $I(1, y) = y$;
I7: $I(x, I(y, z)) = I(y, I(x, z))$;
I8: $I(x, y) = 1 \Leftrightarrow x \leq y$;
I9: $I(x, y) = I(N(y), N(x))$, N is a SFN;
I10: $I(x, y) = 0 \Leftrightarrow x = 1$ and $y = 0$;

If $I : U^2 \to U$ is a fuzzy implication satisfying **I1**, then the function $N_I : U \to U$ defined by

$$N_I(x) = I(x, 0) \tag{2}$$

is a fuzzy negation [3, Lemma 2.1].

Let S be a t-conorm and N be a fuzzy negation. An S-implication [3, 6, 7] is a fuzzy implication $I_{S,N} : U^2 \longrightarrow U$ defined by

$$I_{S,N}(x, y) = S(N(x), y). \tag{3}$$

In this paper, such S-implications are called *strong S-implications*. In [10, Theorem 3.2] $I : U^2 \to U$ is a *strong S-implication* if and only if it satisfies **I1**,**I2**, **I6**, **I7** and **I9**. a characterization of strong S-implications considering **I1**, **I4** and **I7**. Strong S-implications satisfy **I3**, **I4**, **I9** and properties below:

I11: $I(x, y) \geq N_I(x)$; **I12**: $I(x, y) = 0$ if and only if $x = 1$ and $y = 0$.

1.2 Intuitionistic Fuzzy Connectives

According with [1], an intuitionistic fuzzy set (IFS) A_I in a non-empty, universe χ, is expressed as $A_I = \{(x, \mu_A(x), \nu_A(x)) : x \in \chi, \mu_A(x) + \nu_A(x)) \leq 1\}$. Thus, an intuitionistic fuzzy truth value of an element x in an IFS A_I is related to the ordered pair $(\mu_A(x), \nu_A(x))$. Moreover, an IFS A_I generalizes a FS $A = \{(x, \mu_A(x)) : x \in \chi\}$, since $\nu_A(x)$, which means that the non-membership degree of an element x, is less than or equal to the complement of its membership degree $\mu_A(x)$, and therefore $\nu_A(x)$ is not necessarily equal to its complement $1 - \mu_A(x)$.

Let $\tilde{U} = \{(x_1, x_2) \in U^2 | x_1 \leq N_S(x_2)\}$ be the set of all intuitionistic fuzzy values and $l_{\tilde{U}}, r_{\tilde{U}} : \tilde{U} \to U$ be the projection functions on $\tilde{U}$, which are given by $l_{\tilde{U}}(\tilde{x}) = l_{\tilde{U}}(x_1, x_2) = x_1$ and $r_{\tilde{U}}(\tilde{x}) = r_{\tilde{U}}(x_1, x_2) = x_2$, respectively.

Thus, for all $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_n) \in \tilde{U}^n$, such that $\tilde{x}_i = (x_{i1}, x_{i2})$ and $x_{i1} \leq N_S(x_{i2})$ when $1 \leq i \leq n$, considering $l_{\tilde{U}^n}, r_{\tilde{U}^n} : \tilde{U}^n \to U^n$ as the projections given by:

$$l_{\tilde{U}^n}(\tilde{\mathbf{x}}) = (l_{\tilde{U}}(\tilde{x}_1), l_{\tilde{U}}(\tilde{x}_2), \ldots, l_{\tilde{U}}(\tilde{x}_n)) = (x_{11}, x_{21}, \ldots x_{n1}); \tag{4}$$

$$r_{\tilde{U}^n}(\tilde{\mathbf{x}}) = (r_{\tilde{U}}(\tilde{x}_1), r_{\tilde{U}}(\tilde{x}_2), \ldots r_{\tilde{U}}(\tilde{x}_n)) = (x_{12}, x_{22}, \ldots x_{n2}). \tag{5}$$

By [1], for $\tilde{x}, \tilde{y} \in \tilde{U}$, the order relation $\leq_{\tilde{U}}$ is given as $\tilde{x} \leq_{\tilde{U}} \tilde{y} \Leftrightarrow x_1 \leq y_1$ and $x_2 \geq y_2$, such that $\tilde{0} = (0, 1) \leq_{\tilde{U}} \tilde{x}$ and $\tilde{1} = (1, 0) \geq_{\tilde{U}} \tilde{x}$. Moreover, the expression is known:

$$\tilde{x} \preceq_{\tilde{U}} \tilde{y} \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \leq y_2. \tag{6}$$

Additonally, a function $\pi_A : \chi \to U$, called an **intuitionistic fuzzy index** (IFIx) of an element $x \in \chi$, related to an IFS $\mathbf{A}$, is given as

$$\pi_A(x) = N_S(\mu_A(x) + \nu_A(x)), \forall x \in \chi, \mu_A(x) + \nu_A(x) \leq 1. \tag{7}$$

Such function provides the hesitancy (indeterminance) degree of x in A. Based on this, the accuracy function $h_A : \chi \to U$ provides the accuracy degree of x in A, given as:

$$h_A(x) + \pi_A(x) = 1 \tag{8}$$

Therefore, the largest $\pi_A(x)$ ($h_A(x)$), the higher the hesitancy (accuracy) degree of x in A.

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An *intuitionistic fuzzy negation* (IFN shortly) $N_I : \tilde{U} \to \tilde{U}$ satisfies, for all $\tilde{x}, \tilde{y} \in \tilde{U}$, the following properties:
$\mathbf{N}_I\mathbf{1}$: $N_I(\tilde{0}) = N_I(0,1) = \tilde{1}$ and $N_I(\tilde{1}) = N_I(1,0) = \tilde{0}$;
$\mathbf{N}_I\mathbf{2}$: If $\tilde{x} \geq \tilde{y}$ then $N_I(\tilde{x}) \leq N_I(\tilde{y})$.
Additionally, N_I is a **strong intuitionistic fuzzy negation** (SIFN) verifying the condition:
$\mathbf{N}_I\mathbf{3}$: $N_I(N_I(\tilde{x})) = \tilde{x}$, $\forall \tilde{x} \in \tilde{U}$.

Consider N_I as IFN in $\tilde{U}$ and $\tilde{f} : \tilde{U}^n \to \tilde{U}$. For all $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_n) \in \tilde{U}^n$, the **$N_I$-dual intuitionistic function of $\tilde{f}$**, denoted by $\tilde{f}_{N_I} : \tilde{U}^n \to \tilde{U}$, is given by:

$$\tilde{f}_{N_I}(\tilde{\mathbf{x}}) = N_I(\tilde{f}(N_I(\tilde{x}_1), \ldots, N_I(\tilde{x}_n))). \tag{9}$$

When $\tilde{N}_I$ is a SIFN, $\tilde{f}$ is a self-dual intuitionistic function. Additionally, by [2], a SIFN $N_I : \tilde{U} \to \tilde{U}$ is a SIFN iff there exists a SFN $N : U \to U$ such that: $N_I(\tilde{x}) = (N(N_S(x_2)), N_S(N(x_1)))$. Additionally, if $N = N_S$, we have that $N_I(\tilde{x}) = (x_2, x_1)$.

2 Generalized Atanassov's Intuitionistic Fuzzy Index

Definition 2.1. *[5, Definition 1], A function $\Pi : \tilde{U} \to U$ is called a generalized intuitionistic fuzzy index associated with a SFN N ($A - GIFIx(N)$) if, for all $x, y, z, t \in U$, it holds that:*
$\mathbf{\Pi 1}$*: $\Pi(x,y) = 1$ if and only if $x = y = 0$;*
$\mathbf{\Pi 2}$*: $\Pi(x,y) = 0$ if and only if $x + y = 1$;*
$\mathbf{\Pi 3}$*: if $(z,t) \preceq_{\tilde{U}} (x,y)$ then $\Pi(x,y) \leq \Pi(z,t)$;*
$\mathbf{\Pi 4}$*: $\Pi(x,y) = \Pi(N_I(x,y))$ when N_I is a SIFN.*

Proposition 2.1. *[5, Theorem 3] Let N_I be a SFN. A function $\Pi : \tilde{U} \to U$ is a $A - GIFIx(N)$ iff there exists a function $I : U^2 \to U$ verifying* **I1***,* **I8***,***I9** *and* **I10** *such that*

$$\Pi_I(x,y) = N(I(1-y,x)). \tag{10}$$

See Table 2, illustrating Prop. 2.2 by presenting examples of A-GIFIx associated with following fuzzy implications: R_0, Lukaziewicz, Reichenbach, Gaines-Rescher and I_{30} [9].

2.1 A-GIFIx and conjugate fuzzy implications

Proposition 2.2. *Let N_I be a SFN, $\phi \in Aut(U)$ and $I^\phi : U^2 \to U$ be a ϕ-conjugate of $I : U^2 \to U$. A function $\Pi : \tilde{U} \to U$ given by*

$$\Pi_{I^\phi}(x,y) = N^\phi(I^\phi(1-y,x)). \tag{11}$$

is a $A - GIFIx(N)$ whenever $\Pi_I : \tilde{U} \to U$ is also a $A - GIFIx(N)$.

Proof. *($\Rightarrow$) Suppose that $\Pi : \tilde{U} \to U$, $\Pi_I(x,y) = N(I(1-y,x))$, is a $A - GIFIx(N)$. Then, $I, I^\phi : U^2 \to U$ verify* **I1***,* **I8***,***I9** *and* **I10***. For $\Pi_{I^\phi}(x,y) = N^\phi(I^\phi(1-y,x))$ the*

Table 1: Generalized intuitionistic fuzzy index associated with the standard negation.

Fuzzy Implications	$A-GIFIx$
$I_0(x,y)=\begin{cases}1, & \text{if } x\le y,\\ \max(1-x,y), & \text{otherwise;}\end{cases}$	$\Pi_0(x,y)=\begin{cases}0, & \text{if } x+y=1,\\ 1-\max(x,y), & \text{otherwise;}\end{cases}$
$I_{LK}(x,y)=\begin{cases}1, & \text{if } x\le y,\\ 1-x+y, & \text{otherwise;}\end{cases}$	$\Pi_{LK}(x,y)=\begin{cases}0, & \text{if } x+y=1,\\ 1-x-y, & \text{otherwise;}\end{cases}$
$I_{RB}(x,y)=\begin{cases}1, & \text{if } x\le y,\\ 1-x+xy, & \text{otherwise;}\end{cases}$	$\Pi_{RB}(x,y)=\begin{cases}0, & \text{if } x+y=1,\\ 1-x-y+xy, & \text{otherwise;}\end{cases}$
$I_{GR}(x,y)=\begin{cases}1, & \text{if } x\le y,\\ 0, & \text{otherwise;}\end{cases}$	$\Pi_{GR}(x,y)=\begin{cases}0, & \text{if } x+y=1,\\ 1, & \text{otherwise;}\end{cases}$
$I_{30}(x,y)=\begin{cases}\min(1-x,y,0.5), & \text{if } 0<x<y<1,\\ \min(1-x,y), & \text{otherwise;}\end{cases}$	$\Pi_{30}(x,y)=\begin{cases}1-\min(x,y,0.5), & \text{if } 0<x,y<1 \text{ and } x+y=1,\\ 1-\min(x,y), & \text{otherwise;}\end{cases}$

following holds:

$$
\begin{aligned}
\mathbf{\Pi 1}: N^\phi(I^\phi(1-y,x))=1 &\Leftrightarrow I^\phi(1-y,x)=0, \ \ \textit{by } \mathbf{N3}\\
&\Leftrightarrow \phi^{-1}(I(\phi(1-y),\phi(x)))=0, \ \ \textit{by Eq.(1)}\\
&\Leftrightarrow I(\phi(1-y),\phi(x)))=0, \ \ \textit{by } \mathbf{A2}, \mathbf{A3}\\
&\Leftrightarrow \phi(1-y)=1\, and\, \phi(x)=0, \ \ \textit{by } \mathbf{I10}\\
&\Leftrightarrow 1-y=1\, and\, x=0 \Leftrightarrow y=0\, and\, x=0, \ \ \textit{by } \mathbf{A2}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{\Pi 2}: N^\phi(I^\phi(1-y,x))=0 &\Leftrightarrow I^\phi(1-y,x)=1, \ \ \textit{by } \mathbf{N3}\\
&\Leftrightarrow \phi^{-1}(I(\phi(1-y),\phi(x)))=1, \ \ \textit{by Eq.(1)}\\
&\Leftrightarrow I(\phi(1-y),\phi(x)))=1, \ \ \textit{by } \mathbf{A2}, \mathbf{A3}\\
&\Leftrightarrow \phi(1-y)=1\le\phi(x)=0, \ \ \textit{by } \mathbf{I9}\\
&\Leftrightarrow 1-y\le x \Leftrightarrow x+y=1, \ \ \textit{by } \mathbf{A1}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{\Pi 3}: (z,t)\preceq(x,y) &\Rightarrow z\le x\, and\, t\le y, \ \ \textit{by Eq.(6)}\\
&\Rightarrow z\le x\, and\, N_S(t)\le N_S(y), \ \ \textit{by } \mathbf{N2}\\
&\Rightarrow \phi(z)\le\phi(x) and \phi(1-t)\ge\phi(1-y), \ \ \textit{by } \mathbf{A1}\\
&\Rightarrow I^\phi(1-y,x)\ge I^\phi(1-t,z), \ \ \textit{by } \mathbf{I1}\\
&\Rightarrow N(I^\phi(1-y,x))\le N^\phi(I^\phi(1-t,z)), \ \ \textit{by } \mathbf{N2}\\
&\Rightarrow \Pi_{I^\phi}(x,y)\le\Pi_{I^\phi}(z,t), \ \ \textit{by Eq.(11)}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{\Pi 4}: \textit{When } N_I \textit{ is a SIFN, } \Pi(N_I(x,y)) &= \Pi_{I^\phi}(N(N_S(y)),N_S(N(x))), \ \ \textit{by Eq.(2)}\\
&= N^\phi(I^\phi({N_S}^2(N(x)),N(N_S(y))), \ \ \textit{by Eq.(11)}\\
&= N^\phi(I^\phi(N(x),N(N_S(y))), \ \ \textit{by } \mathbf{N3}\\
&= N^\phi(I^\phi(N^2(N_S(y)),N^2(x)), \ \ \textit{by } \mathbf{I9}\\
&= N^\phi I^\phi(I^\phi(N_S(y),x)=\Pi(x,y), \ \ \textit{by } \mathbf{N3}
\end{aligned}
$$

Therefore, Proposition 2.2 holds.

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Table 2: A-GIFIx associated with the automorphisms $\phi(x) = x^2$ *and* $\phi^{-1} = \sqrt{x}$.

Fuzzy Implications	$A - GIFIx$
$I_0^\phi(x,y) = \begin{cases} 1, \text{ if } x \leq y, \\ \sqrt{\max((1-x)^2, y^2)}, \text{otherwise;} \end{cases}$	$\Pi_{I_0^\phi}(x,y) = \begin{cases} 0, \text{ if } x+y=1, \\ 1 - \sqrt{\max(y^2, x^2)}, \text{otherwise;} \end{cases}$
$I_{LK}^\phi(x,y) = \begin{cases} 1, \text{ if } x \leq y, \\ \sqrt{1-x^2+y^2}, \text{ otherwise;} \end{cases}$	$\Pi_{I_{LK}^\phi}(x,y) = \begin{cases} 0, \text{ if } x+y=1, \\ 1 - \sqrt{2y-y^2+x^2}, \text{otherwise;} \end{cases}$
$I_{RB}^\phi(x,y) = \begin{cases} 1, \text{ if } x \leq y, \\ \sqrt{1-x^2+x^2y^2}, \text{otherwise;} \end{cases}$	$\Pi_{I_{RB}^\phi}(x,y) = \begin{cases} 0, \text{ if } x+y=1, \\ 1-\sqrt{x^2+(1-x^2)(2y-y^2)}, \text{otherwise;} \end{cases}$
$I_{GR}^\phi(x,y) = \begin{cases} 1, \text{ if } x \leq y, \\ 0, \text{otherwise;} \end{cases}$	$\Pi_{I_{GR}^\phi}(x,y) = \begin{cases} 0, \text{ if } x+y=1, \\ 1, \text{otherwise;} \end{cases}$
$I_{30}^\phi(x,y) = \begin{cases} \sqrt{\min(1-x^2, y^2, 0.5)}, \\ \quad \text{if } 0 < x < y < 1, \\ \sqrt{\min((1-x)^2, y^2)}, \text{otherwise;} \end{cases}$	$\Pi_{I_{30}^\phi}(x,y) = \begin{cases} 1-\sqrt{\min(1-(1-y)^2, x^2, 0.5)}, \\ \quad \text{if } 0 < x, y < 1 \text{ and } x+y=1, \\ 1-\sqrt{\min(1-(1-y)^2, x^2)}, \text{otherwise;} \end{cases}$

See Table 2.1, presenting the corresponding $A - GIFIx(N)$ associated with the conjugate fuzzy implications related to Table 2:

2.2 A-GIFIx S-implications

By [2], a continuous fuzzy implication I satisfies properties **I7** and **I8** iff it is conjugate with the Lukasiewicz implication (I_{RH}^ϕ) and the following proposition holds:

Proposition 2.3. *[5, Proposition 2] Let ϕ_1, ϕ_2 be automorphisms on U. Then*

$$\Pi_{I_{RH}^\phi}(x) = \phi_1^{-1}(\phi_2(1-x_2) - \phi_2(x1)), \forall x \in U, \quad (12)$$

is a A-GIFIx associated with the SFN associated with a SFN $N(x) = \phi_2^{-1}(1 - \phi_2(x))$.

Proposition 2.4. *Let N_I be a SFN. A function $\Pi : \tilde{U} \to U$ is a $A - GIFIx(N)$ iff there exists an (S,N)-implication $I_{S,N} : U^2 \to U$ such that*

$$\Pi_I(x,y) = N(S(N(1-y), x)). \quad (13)$$

Proof. $\Pi_I(x,y) = N(I_{S,N}(1-y,x)) = N(S(N(1-y),x))$, *for all* $(x,y) \in \tilde{U}$.

Remark 2.1. *When $N = N_S$, Eq.(13) can be expressed as $\Pi_I(x,y) = N_S(S(x,y))$.*

3 Conclusion

In this work, the concept of generalized Atanassov's intuitionistic fuzzy index was studied from different construction methods, in particular, by means of fuzzy S-implication operators and automorphisms. Further work considers the extension of such study related to properties verified by the A-GIFIX to the interval-valued intuitionistic fuzzy approach.

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