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A Geometric Motivated Approach of Lie Derivative of Spinor Fields

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Abstract. In this paper using the Clifford bundle ($\mathcal{C}\ell(M, \mathtt{g})$) and spin-Clifford bundle ($\mathcal{C}\ell_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$) formalism, which permit to give a meaningfull representative of a Dirac-Hestenes spinor field (even section of $\mathcal{C}\ell_{\mathrm{Spin}^e_{1,3}}(M, \mathtt{g})$) in the Clifford bundle, we give a geometrical motivated definition for the Lie derivative of spinor fields in a Lorentzian structure $(M, \boldsymbol{g})$ where M is a manifold such that $\dim M = 4$, $\boldsymbol{g}$ is Lorentzian of signature $(1, 3)$. Our Lie derivative, called the spinor Lie derivative (and denoted $\overset{s}{\pounds}_{\boldsymbol{\xi}}$) is given by nice formulas when applied to Clifford and spinor fields, and moreover $\overset{s}{\pounds}_{\boldsymbol{\xi}}\boldsymbol{g} = 0$ for any vector field $\boldsymbol{\xi}$. With this we compare our definitions and results in [11] with the many others appearing in literature on the subject.

Keywords. Lie Derivative, Spinor Fields, Dirac-Hestenes Spinor Fields, Clifford Fiber Bundle, Spin-Clifford Fiber Bundle

1 Introduction

Lie derivatives of tensor fields are defined once we give the concept of the push forward and pullback mappings (which serves the purpose of defining the image of the tensor field) associated to one-parameter groups of diffeomorphisms generated by vector fields. These concepts are well known and very important in the derivation of conserved currents in physical theories.

It happens that physical theories need also the concept of spinor fields living on a Lorentzian manifold and the question arises as how to define a meaningful image for these objects under a diffeomorphism. There are a lot of different approaches to the subject, as the reader can learn consulting, e.g., [1–4, 6–10, 12, 15].

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We recall [14] that fixing a global spinor basis[4] $\Xi_0(x) = (x, u_0(x))$ for $P_{\mathrm{Spin}^e_{1,3}}(M, \boldsymbol{g})$, and given an algebraic spinor $\boldsymbol{\Psi} \in \sec \mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathrm{g})$, the associated Dirac-Hestenes Spinor Field (DHSF) $\Psi \in \sec \mathcal{C}\ell^{0l}_{\mathrm{Spin}^e_{1,3}}(M, \mathrm{g})$ can be represented in the Clifford bundle by the object

$$\psi_{\Xi_0} \in \sec \mathcal{C}\ell^0(M, \mathrm{g}). \tag{1}$$

Remark 1.1. *When* $\psi_{\Xi_0}\tilde{\psi}_{\Xi_0} \neq 0$ *we can easily show that* ψ_{Ξ_0} *has the following representation*

$$\psi_{\Xi_0} = \rho^{\frac{1}{2}} e^{-\frac{\boldsymbol{\tau_g}\beta}{2}} R, \tag{2}$$

where $\rho, \beta \in \sec \bigwedge^0 T^*M \hookrightarrow \sec \mathcal{C}\ell^0(M, \mathrm{g})$ *and* [13]

$$R = \pm e^{\mathcal{F}} \in \sec \mathrm{Spin}^e_{1,3}(M, \mathrm{g}) \hookrightarrow \sec \mathcal{C}\ell^0(M, \mathrm{g}), \tag{3}$$

with $\mathcal{F} \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}\ell^0(M, \mathrm{g})$.

Let $\boldsymbol{\xi} \in \sec TM$ be a smooth vector field. For any $x \in M$ there exists an unique integral curve of $\boldsymbol{\xi}$, given by $t \mapsto \mathrm{h}(t, x)$, with $x = \mathrm{h}(0, x)$. We recall that for $(t, x) \in I(x) \times \mathbb{M}$ $(I(x) \subset \mathbb{R})$ the mapping h: $(t, x) \mapsto \mathrm{h}(t, x)$ is called the flow of $\boldsymbol{\xi}$. We suppose in what follows that the mappings $\mathrm{h}_t := \mathrm{h}(t,\ ) : M \to M$, $x \mapsto x' = \mathrm{h}_t(x)$ generate a one-parameter group of diffeomorphisms of M (i.e., $I(x) = \mathbb{R}$).

Thus, we see that there exists *no* difficulty in defining the pullback of $\rho^{\frac{1}{2}} e^{-\frac{\boldsymbol{\tau_g}\beta}{2}} e^{\mathcal{F}(x)}$ under h_t (or of more generally, for any $\psi_{\Xi_0} \in \mathcal{C}\ell^0(M, \mathrm{g})$), which will be written as

$$\rho^{\frac{1}{2}}(x'(x)) e^{-\frac{\boldsymbol{\tau}'_{\boldsymbol{gt}}(x)\beta(x'(x))}{2}} e^{\mathcal{F}'_t(x)}. \tag{4}$$

However, we immediately have a

> **Problem**: The object defined by Eq.(4) is of course, a representative in $C\ell^0(M, g)$ of some Dirac-Hestenes spinor field but there is no way to know to which the spinor frame that object is associated.

Thus, we must find another way to define the Lie derivative for spinor fields. Our way, as we will see, is based in a geometric motivated definition for the concept of image of Clifford and spinor fields under diffeomorphisms generated by one-parameter group associated to an arbitrary vector field $\boldsymbol{\xi}$. But, we need first to introduce some results proved in [11], starting with the

Proposition 1.1. *Let* $\pounds_{\boldsymbol{\xi}}$ *denotes the standard Lie derivative of tensor fields. If* $\boldsymbol{\xi}$ *is a Killing vector field then*

$$\pounds_{\boldsymbol{\xi}} \boldsymbol{\gamma}^\alpha = \frac{1}{4}[L(\xi) + d\xi, \boldsymbol{\gamma}^\alpha] \tag{5}$$

$$= \boldsymbol{D}_\xi \boldsymbol{\gamma}^\alpha + \frac{1}{4}[d\xi, \boldsymbol{\gamma}^\alpha] \tag{6}$$

[4]Such a basis must exists according to Geroch Theorem [5].

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$$L(\xi) := \frac{1}{2}(c_{\alpha\kappa\iota} + c_{\kappa\alpha\iota} + c_{\iota\alpha\kappa})\xi^{\kappa}\boldsymbol{\gamma}^{\alpha} \wedge \boldsymbol{\gamma}^{\iota} \tag{7}$$

where $c^{\alpha\cdot\cdot}_{\cdot\kappa\iota}$ *are the structure coefficients of the basis* $\{\boldsymbol{e}_{\alpha}\}$ *dual of* $\{\boldsymbol{\gamma}^{\alpha}\}$.

Remark 1.2. *Moreover, one can easily show using the previous results that when* $\mathcal{C} \in \sec \mathcal{C}\ell(M, \mathrm{g})$ *and* $\boldsymbol{\xi} \in \sec TM$ *is a Killing vector field then*

$$\pounds_{\boldsymbol{\xi}}\mathcal{C} = \eth_{\xi}\mathcal{C} + \frac{1}{4}[\mathbf{S}(\xi), \mathcal{C}]. \tag{8}$$

Indeed, Eq.(8) follows trivially by induction and noting that $\pounds_{\boldsymbol{\xi}}(\mathcal{AB}) = \pounds_{\boldsymbol{\xi}}(\mathcal{A})\mathcal{B} + \mathcal{A}\pounds_{\boldsymbol{\xi}}(\mathcal{B})$, where $\mathcal{A}, \mathcal{B} \in \sec \mathcal{C}\ell(M, \mathrm{g})$, when $\boldsymbol{\xi} \in \sec TM$ is a Killing vector field.

This suggests that $L(\xi)$ should be involved in the definition of the Lie derivative of spinor fields. Based on this, and recalling Eq.(3) we propose that the *spinor lifting* of an integral curve of a generic smooth vector field $\boldsymbol{\xi} \in \sec TM$ to $P_{\mathrm{Spin}^{e}1,3}(M, \mathrm{g})$ in the parallelizabe manifold M equipped with the global orthonormal cobasis $\{\boldsymbol{\gamma}^{\alpha}\}$ is given by the following

Definition 1.1. *Consider the integral curve* $\mathrm{h}_t : \mathbb{R} \to M$ *of an arbitrary smooth vector field* $\boldsymbol{\xi}$. *The spinor lifiting* $\breve{h}_t$ *of* h_t *to* $P_{\mathrm{Spin}^{e}_{1,3}}(M, \mathrm{g})$ *is the curve*

$$\breve{h}_t(p) = (h_t(\pi(p)), a u_t(h_t(\pi(p)))) \tag{9}$$

$$u_t(x) := e^{-\frac{1}{4}t(\mathbf{S}(\xi)(x))} \in \mathrm{Spin}^{e}_{1,3}, \tag{10}$$

$$\mathbf{S}(\xi) = L(\xi) + d\xi, \tag{11}$$

with $\pi(p) = \pi((x, a)) = x$.

To see why the above definition is really important consider that for $t << 1$ it is

$$u_t = 1 - \frac{1}{4}t\mathbf{S}(\xi) + O(t^2) + \cdots \tag{12}$$

Then, we have for $t << 1$ that

$$\begin{aligned} u_t^{-1}\boldsymbol{\gamma}^{\alpha}u_t &= \{1 + \frac{1}{4}t\mathbf{S}(\xi) + O(t^2) + \cdots\}\boldsymbol{\gamma}^{\alpha}\{1 - \frac{1}{4}t\mathbf{S}(\xi) + O(t^2) + \cdots\} \\ &= \boldsymbol{\gamma}^{\alpha} + \frac{1}{4}t[\mathbf{S}(\xi), \boldsymbol{\gamma}^{\alpha}] + O(t^2) + \cdots \end{aligned} \tag{13}$$

Deriving in $t = 0$ we obtain the expression of the previous proposition.

Now, recall that the pullback $\boldsymbol{\gamma}_t'^{\alpha} = \mathrm{h}_t^{*}\boldsymbol{\gamma}^{\alpha}$ when $\boldsymbol{\xi}$ is an arbitrary vector field for $t << 1$

$$\boldsymbol{\gamma}_t'^{\alpha}(x) = \boldsymbol{\gamma}^{\alpha}(x) + t\pounds_{\boldsymbol{\xi}}\boldsymbol{\gamma}^{\alpha}(x) + O(t^2) + \cdots \tag{14}$$

Using the Proposition (1.1), comparing Eq.(14) with Eq.(13) and recalling Eq.(5), we see that up to the *first order* we have

$$\boldsymbol{\gamma}_t'^{\alpha}(x) = u_t^{-1}(x)\boldsymbol{\gamma}^{\alpha}(x)u_t(x) \tag{15}$$

From Eq.(15), the Lie derivative $\pounds_{\boldsymbol{\xi}}\boldsymbol{\gamma}^{\alpha}$ can be calculated in two ways, using the usual definition by pullback or by the action of u_t. Note that the action of u_t is always orthogonal, regardless of $\boldsymbol{\xi}$ be Killing. We will use this fact to give our geometric motivated concept of Lie derivatives for Clifford and spinor fields.

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2 The Spinor Lie Derivative $\overset{s}{\pounds}_{\boldsymbol{\xi}}$

2.1 Spinor Images of Clifford and Spinor Fields

Given the spinorial frame $\boldsymbol{\Xi}_{u_t}(x) = (x, u_t)$ in $P_{\mathrm{Spin}^e_{1,3}}(M, \mathbf{g})$ we see that the basis $\{\check{\boldsymbol{\gamma}}^{\alpha}_t\}$ of $P_{\mathrm{SO}^e_{1,3}}(M, \mathbf{g})$ such that

$$\check{\boldsymbol{\gamma}}^{\alpha}_t(x) = u_t^{-1}(x)\boldsymbol{\gamma}^{\alpha}(x))u_t(x) = \Lambda^{\alpha}_{t\beta}(x)\boldsymbol{\gamma}^{\beta}(x), \tag{16}$$

is always orthonormal relative to $\mathbf{g}$. This suggests to define a mapping ${}^s\mathrm{h}_t$ (associated with a one parameter group of diffeomorphisms h_t generated by a vector field $\boldsymbol{\xi}$) acting on sections $\bigwedge^p T^*M$, $\mathcal{C}\ell(M, \mathrm{g})$, $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathbf{g})$, $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M, \mathbf{g})$. With $x' = \mathrm{h}_t(x)$ we start giving

Definition 2.1.

$$\begin{aligned} {}^s\mathrm{h}_t : \sec\mathcal{C}\ell(M, \mathrm{g}) \hookleftarrow \sec\textstyle\bigwedge^p T^*M \to \sec\textstyle\bigwedge^p T^*M \hookrightarrow \sec\mathcal{C}\ell(M, \mathrm{g}), \\ P(x') \mapsto \check{P}_t(x) = \frac{1}{p!}P_{i_1\cdots i_p}(x'(x))\check{\boldsymbol{\gamma}}^{i_1}_t(x)\cdots\check{\boldsymbol{\gamma}}^{i_p}_t(x) \\ = \frac{1}{p!}P_{i_1\cdots i_p}(x'(x))u_t^{-1}\boldsymbol{\gamma}^{i_1}(x)\cdots\boldsymbol{\gamma}^{i_p}(x)u_t \end{aligned} \tag{17}$$

$$\begin{aligned} P(x) &= \frac{1}{p!}P_{i_1\cdots i_p}(x)\boldsymbol{\gamma}^{i_1}(x)\cdots\boldsymbol{\gamma}^{i_p}(x) \neq \check{P}_t(x) \\ P(x') &= \frac{1}{p!}P_{i_1\cdots i_p}(x')\boldsymbol{\gamma}^{i_1}(x')\cdots\boldsymbol{\gamma}^{i_p}(x') \end{aligned} \tag{18}$$

Eq.(17) *extends by linearity to all sections of* $\mathcal{C}\ell(M, \mathrm{g})$. *Given any* $\mathcal{C} \in \sec\mathcal{C}\ell(M, \mathrm{g})$ *we will call* $\check{\mathcal{C}}_t$ *the spinor image of* $\mathcal{C}$.

2.2 Spinor Derivative of Clifford and Spinor Fields

Definition 2.2. *The spinor Lie Derivative* $\overset{s}{\pounds}_{\boldsymbol{\xi}}$ *of a Clifford field* $\boldsymbol{C} = [(\Xi_0, \mathcal{C})]$ *(a section of* $\mathcal{C}\ell(M, \mathrm{g})$*)* in the direction of an arbitrary vector field $\boldsymbol{\xi}$ is

$$\overset{s}{\pounds}_{\boldsymbol{\xi}}\,\boldsymbol{C} = \left.\frac{d}{dt}\check{\boldsymbol{C}}_t\right|_{t=0} \tag{19}$$

A trivial calculation gives

$$\overset{s}{\pounds}_{\boldsymbol{\xi}}\boldsymbol{C} = \mathfrak{d}_{\boldsymbol{\xi}}\,\boldsymbol{C} + \frac{1}{4}[\boldsymbol{S}(\boldsymbol{\xi}), \boldsymbol{C}]. \tag{20}$$

Given that a left DHSF Ψ, a section of $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M, \mathbf{g})$ (respectively Φ, a section of $\mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M, \mathbf{g})$) can be written as

$$\boldsymbol{\psi}_{\Xi_0} = \Psi\mathbf{1}^r_{\Xi_0}, \quad \boldsymbol{\phi}_{\Xi_0} = \mathbf{1}^l_{\Xi_0}\Phi, \tag{21}$$

$$\boldsymbol{\psi}_{\Xi_0}\mathbf{1}^l_{\Xi_0} = \Psi\mathbf{1}^r_{\Xi_0}\mathbf{1}^l_{\Xi_0} = \Psi 1, \quad \mathbf{1}^r_{\Xi_0}\boldsymbol{\phi}_{\Xi_0} = \mathbf{1}^r_{\Xi_0}\mathbf{1}^l_{\Xi_0}\Phi = 1\Phi, \tag{22}$$

$$\Psi = \boldsymbol{\psi}_{\Xi_0}\mathbf{1}^l_{\Xi_0} = \boldsymbol{\psi}_{\Xi_u}\mathbf{1}^l_{\Xi u}, \quad \Phi = \mathbf{1}^r_{\Xi_0}\boldsymbol{\phi}_{\Xi_0} = \mathbf{1}^r_{\Xi_u}\boldsymbol{\phi}_{\Xi_u}. \tag{23}$$

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Using Eq.(23) and that $\boldsymbol{\psi}_{\Xi_0}, \boldsymbol{\phi}_{\Xi_0} \in \sec \mathcal{C}\ell^0(M, \mathrm{g})$, where we know how to act, we propose the following definition:

Definition 2.3. *The spinor images of Ψ and Φ are:*

$$ {}^s\mathrm{h}_t\ \Psi \circ \mathrm{h}_t x = {}^s\Psi_t(x) := (\boldsymbol{u}_t^{-1}\boldsymbol{\psi}_{\Xi_0}(x)\boldsymbol{u}_t)\ {}^s\mathbf{1}^l_{t\Xi_0}, \tag{24}$$

$$ {}^s\mathrm{h}_t \Phi \circ \mathrm{h}_t x = {}^s\Phi_t(\boldsymbol{x}) := {}^s\mathbf{1}^r_{t\Xi_0}(\boldsymbol{u}_t^{-1}\boldsymbol{\phi}_{\Xi_0}(x)\boldsymbol{u}_t) \tag{25}$$

$$ {}^s\mathbf{1}^l_{t\Xi_0} := \boldsymbol{u}_t^{-1}\mathbf{1}^l_{\Xi_0}, \qquad {}^s\mathbf{1}^r_{t\Xi_0} = \mathbf{1}^r_{\Xi_0} u_t. \tag{26}$$

Definition 2.4. *With these actions, we define:*

$$\begin{aligned} \overset{s}{\pounds}_{\boldsymbol{\xi}}\Psi &:= \frac{d}{dt}\ {}^s\Psi_t(x)|_{t=0}\,, \\ \overset{s}{\pounds}_{\boldsymbol{\xi}}\Phi &:= \frac{d}{dt}\ {}^s\Phi_t(x)|_{t=0} \end{aligned} \tag{27}$$

The objects ${}^s\Psi_t, {}^s\boldsymbol{\psi}_{t\Xi_0}, {}^s\Phi_t, {}^s\boldsymbol{\phi}_{t\Xi_0}, {}^s\boldsymbol{C}_t$ (sections of $\mathcal{C}\ell^l_{\mathrm{Spin}^e_{1,3}}(M,\mathrm{g}), \mathcal{C}\ell^r_{\mathrm{Spin}^e_{1,3}}(M,\mathrm{g}), \mathcal{C}\ell(M,\mathrm{g})$) will be referred in what follows as the *spinor images* of the fields $\Psi, \boldsymbol{\psi}_{\Xi_0}, \Phi, \boldsymbol{\phi}_{\Xi_0}, \boldsymbol{C}$.

A trivial calculation gives

$$\begin{aligned} \overset{\boldsymbol{s}}{\pounds}_{\boldsymbol{\xi}}\Psi &= \mathfrak{d}_\xi\Psi + \frac{1}{4}\boldsymbol{S}(\xi)\Psi, \\ \overset{\boldsymbol{s}}{\pounds}_{\boldsymbol{\xi}}\Phi &= \mathfrak{d}_\xi\Phi - \Phi\frac{1}{4}\boldsymbol{S}(\xi) \end{aligned} \tag{28}$$

Remark 2.1. *In the Clifford bundle in the basis Ξ_0, $\psi_{\Xi_0} \in \sec \mathcal{C}\ell(M,\mathrm{g})$ is the representative of $\Psi \in \sec \mathcal{C}\ell^\ell_{\mathrm{Spin}^e_{1,3}}(M,\mathrm{g})$ and if we calculated its spinor Lie derivative as a section of $\mathcal{C}\ell(M,\mathrm{g})$ we should get, of course*

$$ \overset{s}{\pounds}_{\boldsymbol{\xi}}\psi_{\boldsymbol{\Xi}_0} = \mathfrak{d}_{\boldsymbol{\xi}}\psi_{\boldsymbol{\Xi}_0} + \frac{1}{4}[L(\xi) + d\xi, \psi_{\boldsymbol{\Xi}_0}]. \tag{29}$$

This does not mimics the spinor Lie derivative of a DHSF Ψ. Since one of the main reasons to introduce representatives in the Clifford bundle of Dirac-Hestenes spinor fields is to have an easy computation tool when using these representatives together with other Clifford fields we will agree to take as the Lie derivative of ψ_{Ξ_0} an effective Lie derivative denoted $\overset{(s)}{\pounds}_{\boldsymbol{\xi}}\psi_{\boldsymbol{\Xi}_0}$ where the pullback of $\psi_{\boldsymbol{\Xi}_0}$ is the formula given by Eq.(4)*. Thus,*

$$ \overset{(s)}{\pounds}_{\boldsymbol{\xi}}\psi_{\boldsymbol{\Xi}_0} = \mathfrak{d}_{\boldsymbol{\xi}}\psi_{\boldsymbol{\Xi}_0} + \frac{1}{4}L(\xi)\psi_{\boldsymbol{\Xi}_0} + \frac{1}{4}d\xi\psi_{\boldsymbol{\Xi}_0} \tag{30}$$

We then write for $\mathcal{C} \in \sec \mathcal{C}\ell(M,\mathrm{g})$ and $\psi_{\boldsymbol{\Xi}_0}$ as just defined

$$ \overset{(s)}{\pounds}_{\boldsymbol{\xi}}(\mathcal{C}\psi_{\boldsymbol{\Xi}_0}) = (\overset{s}{\pounds}_{\boldsymbol{\xi}}\mathcal{C})\psi_{\boldsymbol{\Xi}_0}) + \mathcal{C}(\overset{(s)}{\pounds}_{\boldsymbol{\xi}}\psi_{\boldsymbol{\Xi}_0}). \tag{31}$$

Remark 2.2. *An analogous concept to* $\overset{(s)}{\pounds}_{\boldsymbol{\xi}}$ *has been introduced in* [14] *for the covariant derivative of representatives in the Clifford bundle of Dirac-Hestenes spinor fields and we recall that for* $\mathcal{C} \in \sec \mathcal{C}\ell(M, \mathrm{g})$ *and* ψ_{Ξ_0} *as above defined we have*

$$\begin{aligned} \overset{(s)}{\boldsymbol{D}}_{\boldsymbol{\xi}}(\mathcal{C}\psi_{\Xi_0}) &= (\boldsymbol{D}_{\boldsymbol{\xi}}\mathcal{C})\psi_{\Xi_0} + \mathcal{C}(\overset{(s)}{\boldsymbol{D}}_{\boldsymbol{\xi}}\psi_{\Xi_0}), \\ \boldsymbol{D}_{\boldsymbol{\xi}}\mathcal{C} &= \mathfrak{d}_{\xi}\mathcal{C} + \frac{1}{2}[\omega_{\xi}, \mathcal{C}], \\ \overset{(s)}{\boldsymbol{D}}_{\boldsymbol{\xi}}\psi_{\Xi_0} &= \mathfrak{d}_{\xi}\psi_{\Xi_0} + \frac{1}{2}\omega_{\xi}\psi_{\Xi_0}. \end{aligned} \tag{32}$$

with $\omega_{\xi} := \frac{1}{2}\xi^{\kappa}\omega_{\alpha\kappa\beta}\boldsymbol{\gamma}^{\alpha}\boldsymbol{\gamma}^{\beta}$ *the called "connection 2-form". Henceforth, to simplify the notation, the covariant derivative acting in a representative in the Clifford bundle of a DHSF will be written as*

$$\boldsymbol{D}^{s}_{\boldsymbol{\xi}}\psi_{\Xi_0} = \mathfrak{d}_{\xi}\psi_{\Xi_0} + \frac{1}{2}\omega_{\xi}\psi_{\Xi_0}$$

and we will write also $\overset{s}{\pounds}_{\boldsymbol{\xi}}\psi_{\Xi_0}$ *(given by* Eq.(31)*) instead of* $\overset{(s)}{\pounds}_{\boldsymbol{\xi}}\psi_{\Xi_0}$.

Remark 2.3. *One can easily verify that with this agreement we have a perfectly consistent formalism. Indeed, recalling that the spinor bundles are modules over* $\mathcal{C}\ell(M, \mathrm{g})$ *and that any section* $\boldsymbol{C}$ *of* $\mathcal{C}\ell(M, \mathrm{g})$ [14] *can be written as the product of a section* Ψ *of* $\mathcal{C}\ell^{l}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathrm{g})$ *by a section* Φ *of* $\mathcal{C}\ell^{r}_{\mathrm{Spin}^{e}_{1,3}}(M, \mathrm{g})$*, i.e.,* $\boldsymbol{C} = \Psi\Phi$ *we immediately verify that the operator* $\overset{s}{\pounds}_{\boldsymbol{\xi}}$ *satisfies when applied to Clifford and spinor fields the Leibniz rule, i.e.,*

$$\overset{s}{\pounds}_{\boldsymbol{\xi}}(\Psi\Phi) = (\overset{s}{\pounds}_{\boldsymbol{\xi}}\Psi)\Phi + \Psi(\overset{s}{\pounds}_{\boldsymbol{\xi}}\Phi), \tag{33}$$

$$\overset{s}{\pounds}_{\boldsymbol{\xi}}(\boldsymbol{C}\Psi) = (\overset{s}{\pounds}_{\boldsymbol{\xi}}\boldsymbol{C})\Psi + \boldsymbol{C}(\overset{s}{\pounds}_{\boldsymbol{\xi}}\Psi), \tag{34}$$

$$\overset{s}{\pounds}_{\boldsymbol{\xi}}(\Phi\boldsymbol{C}) = (\overset{s}{\pounds}_{\boldsymbol{\xi}}\boldsymbol{C})\Phi + \boldsymbol{C}(\overset{s}{\pounds}_{\boldsymbol{\xi}}\Phi). \tag{35}$$

3 Conclusions

Here we claim to have given a geometrical motivated definition for a Lie derivative of spinor fields in a Lorentzian structure $(M, \boldsymbol{g})$, by finding an appropriated image for Clifford and spinor fields under a diffeomorphism generated by an arbitrary vector field $\boldsymbol{\xi}$. We called such operator the *spinor Lie derivative*, denoted $\overset{s}{\pounds}_{\boldsymbol{\xi}}$ which is such that $\overset{s}{\pounds}_{\boldsymbol{\xi}}\boldsymbol{g} = 0$ for arbitrary vector field $\boldsymbol{\xi}$. We compared our definitions and results in [11] with the many others appearing in literature on the subject.

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