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Geodesics and Constant Angular Momentum in the de Sitter Manifold

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Abstract. Let \( M = SO(1,4)/SO(1,3) \cong S^3 \times \mathbb{R} \) (a parallelizable manifold) be a submanifold in the structure \((\mathcal{M}, \mathcal{g})\) (hereafter called the bulk) where \( \mathcal{M} \simeq \mathbb{R}^5 \) and \( \hat{g} \) is a pseudo Euclidean metric of signature \((1,4)\). Let \( i : M \to \mathbb{R}^5 \) be the inclusion map and let \( g = i^* \hat{g} \) be the pullback metric on \( M \). It has signature \((1,3)\). Let \( D \) be the Levi-Civita connection of \( g \). We call the structure \((M, g)\) a de Sitter manifold and \( M^{dSL} = (M, g, D, \tau_g, \uparrow) \) a de Sitter spacetime structure, which is of course orientable by \( \tau_g \in \text{sec} \bigwedge^4 T^* M \) and time orientable (by \( \uparrow \)). Under these conditions, here we want to present the results that appears in [5–7] in particular that if the motion of a free particle moving on \( M \) happens with constant bulk angular momentum then its motion in the structure \( M^{dSL} \) is a timelike geodesic. Also any geodesic motion in the structure \( M^{dSL} \) implies that the particle has constant angular momentum in the bulk. So using the Clifford and spin-Clifford formalisms [3] and the natural hypothesis that a particle moving freely in \((M, g)\) has constant bulk angular momentum leads naturally to the Dirac equation as found in [1] in the de Sitter structure \((M, g)\).

Keywords. de Sitter Manifold, Geodesics, Angular Momentum, General Relativity

1 Introduction

In what follows \( SO(1,4) \) and \( SO(1,3) \) denote the special pseudo-orthogonal groups in \( \mathbb{R}^{1,4} = (M = \mathbb{R}^5, \hat{g}) \) where \( \hat{g} \) is a metric of signature \((1,4)\). The de Sitter manifold \( M \) can be viewed as a brane (a submanifold) in the structure \( \mathbb{R}^{1,4} \). The structure \( M^{dSL} = (M, g, D, \tau_g, \uparrow) \) will be called Lorentzian de Sitter spacetime structure where, if \( i : \mathbb{R} \times S^3 \to \mathbb{R}^5 \) is the inclusion mapping, \( g = i^* \hat{g} \) and \( D \) is the parallel projection on \( M \) of the pseudo Euclidian metric compatible connection \( \hat{D} \) in \( \mathbb{R}^{1,4} \) (details in [4, 5]). As well known, \((M, g)\), a pseudo-sphere is a spacetime of constant Riemannian curvature. It has ten Killing vector fields. The Killing vector fields are the generators of infinitesimal actions of the group \( SO(1,4) \) (called the de Sitter group) in \( M \). The group \( SO(1,4) \) acts transitively in \( SO(1,4)/SO(1,3) \), which is thus a homogeneous space (for \( SO(1,4) \)).

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The structure $M^{dSL}$ has been used by many physicists as an alternative arena for the motion of particles and fields in place of the Minkowski spacetime structure\(^3\) $\mathfrak{M}$. One of the reasons is that the isometry group of the structure $(M,g)$ is the de Sitter group, which as well known reduces to the Poincaré group when he radius $\ell$ of $(M,g)$ goes to $\infty$. Now, as well known the natural motion of a free particle of mass $m$ in $\mathfrak{M}$ occurs with constant momentum $p = m\pi$, where $\pi : \mathbb{R} \rightarrow \mathcal{M}$ is a timelike curve pointing to the future. The question which naturally arises is the following:

*Which is the natural motion of a free particle of mass $m$ in the structure $(M,g)$?*

One natural suggestion given the well known relation between the de Sitter and Poincaré groups [2] is that such a motion occurs with constant angular momentum $L$ as determined by (hyper observers) living in the bulk. Given this hypothesis we proved in [6] the following proposition: (a): If a particle travels with geodesic motion in the structure $M^{dSL}$ then its bulk angular momentum $L$ is constant. (b): Also, if a particle of mass $m$ constrained to move in $M$ occurs with constant bulk angular $L$ then its motion for an observer living in the brane $M$ is described by a timelike geodesic in the structure $M^{dSL}$.

2 The Lorentzian de Sitter $M^{dSL}$ Structure and its (Projective) Conformal Representation

Let $SO(1,4)$ and $SO(1,3)$ be respectively the special pseudo-orthogonal groups in the structures $\mathbb{R}^{1,4} = \{M = \mathbb{R}^5, \tilde{g}\}$ and $\mathbb{R}^{1,3} = \{\mathbb{R}^4, \eta\}$ where $\tilde{g}$ is a metric of signature $(1,4)$ and $\eta$ a metric of signature $(1,3)$. The manifold $M = SO(1,4)/SO(1,3)$ will be called the *de Sitter manifold*. Since

$$M = SO(1,4)/SO(1,3) \approx SO(1,4)/SO(1,3) \approx \mathbb{R} \times S^3$$

(1)

this manifold can be viewed as a brane [4] (a submanifold) in the structure $\mathbb{R}^{1,4}$. In General Relativity studies it is introduced a Lorentzian spacetime, i.e., the structure $M^{dSL} = (M = \mathbb{R} \times S^3, g, D, \tau_g, \uparrow)$ called *Lorentzian de Sitter spacetime structure*\(^4\) where if $\iota : \mathbb{R} \times S^3 \rightarrow \mathbb{R}^5$ is the inclusion mapping, $g := \iota^*\tilde{g}$ and $D$ is the parallel projection on $M$ of the pseudo Euclidian metric compatible connection in $\mathbb{R}^{1,4}$ (details in [5]). As well known, $M^{dSL}$ is a spacetime of constant Riemannian curvature. It has ten Killing vector fields. The Killing vector fields are the generators of infinitesimal actions of the group $SO(1,4)$ (called the de Sitter group) in $M = \mathbb{R} \times S^3 \approx SO(1,4)/SO(1,3)$. The group $SO(1,4)$ acts transitively in $SO(1,4)/SO(1,3)$, which is thus a homogeneous space (for $SO(1,4)$).

We now give a description of the manifold $\mathbb{R} \times S^3$ as a pseudo-sphere (a submanifold) of radius $\ell$ of the pseudo Euclidean space $\mathbb{R}^{1,4} = \{\mathbb{R}^5, \tilde{g}\}$. If $(X^1, X^2, X^3, X^4, X^0)$ are the

\(^3\)Minkowski spacetime is the structure $\mathfrak{M} = (M = \mathbb{R}^4, \eta, D, \tau_\eta, \uparrow)$ where $\eta$ is the usual Minkowski metric, $\tau_\eta \in \wedge^4 T^*\mathcal{M}$ defines an orientation and $\uparrow$ denotes that $(M, \eta)$ is time orientable. Details in [3].

\(^4\)It is a vacuum solution of Einstein equation with a cosmological constant term. We are not going to use this structure in this paper.
global orthogonal coordinates of \( \mathbb{R}^{1,4} \), then the equation representing the pseudo sphere is

\[
(X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 - (X^4)^2 = -\ell^2.
\]  

(2)

Introducing projective conformal coordinates \( \{x^\mu\} \) by projecting the points of \( \mathbb{R} \times S^3 \) from the “north-pole” to a plane tangent to the “south pole” we see immediately that \( \{x^\mu\} \) covers all \( \mathbb{R} \times S^3 \) except the “north-pole”. We have [2,5,7,8]

\[
X^\mu = \Omega x^\mu, \quad X^4 = -\ell \Omega \left( 1 + \frac{\sigma^2}{4\ell^2} \right)
\]

(3)

with

\[
\Omega = \left( 1 - \frac{\sigma^2}{4\ell^2} \right)^{-1}, \quad \sigma^2 = \eta_{\mu\nu} x^\mu x^\nu
\]

(4)

and we immediately find that

\[
g := \iota^* \hat{g} = \Omega^2 \eta_{\mu\nu} dx^\mu \otimes dx^\nu,
\]

(5)

and the matrix with entries \( \eta_{\mu\nu} \) is the diagonal matrix \( \text{diag}(1,-1,-1,-1) \).

### 3 Constant Bulk Angular Momentum versus Geodesic Equation

Now, write \( D_{\alpha\nu} \partial_\nu = \Gamma^\alpha_{\mu\nu} \partial_\alpha \partial_\nu \) and let \( \sigma : I \to M, s \mapsto \sigma(s) \) be a time like geodesic in \( M \). Its tangent vector field \( \sigma_s \) such that \( \sigma_s(s) = \frac{dx^\mu(\sigma(s))}{ds} \partial s |_{\sigma} = \frac{dx^\nu}{ds} \partial_{x^\nu} \) satisfy \( D_{\sigma_s} x^\mu = 0 \) and in components it is

\[
\frac{d^2 x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0.
\]

(6)

In [6] we obtain the following equation for this geodesic in the de Sitter manifold:

\[
\frac{d^2 x^\alpha}{ds^2} + \frac{\Omega}{\ell^2} x^\mu \frac{dx^\mu}{ds} \frac{dx^\alpha}{ds} - \frac{\Omega}{2\ell^2} x^\delta \frac{dx_\delta}{ds} \frac{dx^\mu}{ds} = 0.
\]

(7)

Let \( \{E_A = \frac{\partial}{\partial x^A}\} \), \( A = 0, 1, 2, 3, 4 \) be the canonical basis of \( T^* M = T^* \mathbb{R}^5 \) and let \( \{E^A = dx^A\} \) be a basis of \( T^* M \) dual to \( \{E_A = \frac{\partial}{\partial x^A}\} \). We have

\[
\hat{g} = \eta_{AB} E^A \otimes E^B
\]

(8)

where the matrix with entries \( \eta_{AB} \) is the diagonal matrix \( \text{diag}(1,-1,-1,-1,-1) \). Moreover let \( \hat{g} = \eta^{AB} E_A \otimes E_B \) be the metric of the cotangent bundle (with \( \eta^{AC} \eta_{CB} = \delta^A_B \)). Finally let \( \{E_A\} \) be the reciprocal basis of \( \{E^A\} \), i.e., \( \hat{g}(E^A, E_B) = \delta^A_B \). We introduce the basis \( \{E_A\} \) of \( \mathbb{R}^5 \) and make the usual identification \( E_A(p) \approx E_A(p') = \mathcal{E}_A, E_A(p) \simeq E_A(p') = \mathcal{E}_A \) for any \( p, p' \in \mathbb{R}^5 \).

Let \( \mathbf{X} = X^A \mathcal{E}_A \) be the position covector, \( \mathbf{P} = m \bar{X} B \mathcal{E}_B \) the bulk momentum covector and \( \mathbf{L} = \mathbf{X} \wedge \mathbf{P} \) the bulk angular momentum of a particle of mass \( m \) in the bulk spacetime
If the particle is constrained to move "freely" in the submanifold \( \mathbb{R} \times S^3 \) a natural hypothesis is that its bulk angular momentum is a constant of motion. Now, \( L = cte \) implies immediately
\[
\frac{1}{2} (X^A \ddot{X}^B - \ddot{X}^A X^B) E_A \wedge E_B = 0. 
\]
(9)

Thus, for \( \kappa, \iota \neq 0, 1, 2, 3 \) it is \( X^\kappa \ddot{X}^\iota - \ddot{X}^\kappa X^\iota = 0 \) and \( X^\kappa \ddot{X}^4 - \ddot{X}^\kappa X^4 = 0 \), so when we use the conformal coordinates we get [6]:
\[
x^k \left( \frac{dx^i}{ds} \frac{1}{l^2} \Omega^2 x_i \frac{dx^j}{ds} + \Omega \frac{d^2 x^j}{ds^2} \right) - \left( \frac{dx^i}{ds} \frac{1}{l^2} \Omega^2 x_i \frac{dx^k}{ds} + \Omega \frac{d^2 x^k}{ds^2} \right) = 0 \tag{10}
\]
\[
(2\Omega - 1) \frac{d^2 x^k}{ds^2} + \frac{1}{l^2} \Omega (2\Omega - 1) x_i \frac{dx^i}{ds} \frac{dx^k}{ds} - \frac{1}{2l^4} \Omega^2 x_i x_j x^k \frac{dx^i}{ds} \frac{dx^j}{ds} - \frac{1}{2l^2} \Omega x_k \frac{dx_i}{ds} \frac{dx^i}{ds} - \frac{1}{2l^2} \Omega x_i \frac{d^2 x^i}{ds^2} x^k = 0, \tag{11}
\]
which are the equations of motion according to the structure \( M^{dSL} \).

With this notations and hypotheses we have proved in [6] the following proposition:

**Proposition 3.1.** (a): If a particle travels with geodesic motion in the structure \( M^{dSL} \) then its bulk angular momentum \( L \) is constant. (b): Also, if a particle of mass \( m \) constrained to move in \( M \) occurs with constant bulk angular \( L \) then its motion for an observer living in the brane \( M \) is described by a timelike geodesic in the structure \( M^{dSL} \).

### 4 Conclusions

We said in the introduction that the de Sitter structure \( M^{dSL} \) has been studied by many authors as a possible natural arena for the motion of particles and fields instead of the Minkowski spacetime structure \( \mathfrak{M} \). We discussed these issues in [5]. At least we want to emphasize that recently it has been shown in [7] by using the Clifford and spin-Clifford formalisms [3] that the hypothesis that a particle moving freely in \((M, g)\) has constant bulk angular momentum leads naturally to the Dirac equation as found in [1] in the de Sitter structure \((M, g)\).

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### Referências


\(^5\) From a physical point of view the statement moving 'freely' means that observers living in \( M \) cannot detect any force acting on the particle.


