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New exact general solutions of a natural extension of the Abel's equation of the second kind

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1 Introduction

The description problem of integrable cases of the Abel's polynomial differential equations remains in focus of current researches. As far as we know, there exist results for generating exact general solutions for the general form of the Abel's equation of the second kind (see [1]), but the same does not occur for the general form of this equation with term of cubic nonlinearity of the type $f_3(x)z^3$, where $f_3(x) \in C(a,b)$ and $z = z(x) \in C^1(a,b)$. In this paper, we introduce a new theorem, which under an existence condition, allows the construction of exact general solutions for the Abel's equation of the second kind with the extension $f_3(x)z^3$, where $f_3(x)$ is an arbitrary continuous function. This result improves and generalizes earlier results from literature.

Several natural phenomena can be modeled, by using the classes of Abel's nonlinear ordinary differential equations. For example, there exist direct applications on nonlinear mechanics (Duffing's oscillator and Van der Pol's oscillator), theory of chemical reactors and combustion theory [2](1.3.5-2) and relativistic dissipative cosmological model. Here the notation $\cdot'_x = \frac{d\cdot}{dx}$ denotes the classical derivative with respect to the independent variable x.

A natural extension of the Abel's equation of the second kind is given by

$$[g_0(x) + g_1(x)z]\, z'_x = \sum_{n=0}^{3} f_n(x)z^n, \tag{1}$$

satisfying $z \in C^1(a,b)$, $g_0(x)$, $g_1(x)$, $f_n(x) \in C(a,b)$ with $n = 0, 1, 2, 3$. We know that equation (1) can be reduced to the Abel's normal form $yy'_x - y = F(x)$, an equivalent equation that does not accept exact general solutions in terms of known functions for $F(x)$ arbitrary. Moreover, only very restricted cases of equation (1) are solvable by means of parametric forms (see for instance [3], [2]). In the present paper, we go a step beyond because we consider a more general case, by introducing a new direct analytical method for obtaining exact general solutions for the general form of equation (1) with $g_0(x), g_1(x), f_3(x) \neq 0$ and $f_0(x) = 0$. For this, we present a new theorem that has a constructive demonstration such that we can find these solutions by means of the proof's idea. We use a new functional relation between the variable coefficients of equation (1), an appropriate and admissible functional transformation and an argument of integrating factors.

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2 Main Result

Theorem 2.1. *For the general form of equation (1) with $f_3(x) \neq 0$ and $f_0(x) = 0$, if there exists a constant γ such that*

$$2\mu_1(x)g_1(x) = \gamma\mu_2(x)g_0(x), \quad g_i(x) \neq 0 \quad i = 1, 2 \tag{2}$$

then equation (1) has the exact implicit general solution

$$\mu_1(x) + \gamma\mu_2(x)z + z^2\left(2\int \mu_1(x)\frac{f_3(x)}{g_0(x)}\,dx + C\right) = 0, \tag{3}$$

where $\mu_1(x) = exp\left(2\int \frac{f_1(x)}{g_0(x)}\,dx\right)$, $\mu_2(x) = exp\left(\int \frac{f_2(x)}{g_1(x)}\,dx\right)$ are integrating factors and C is an arbitrary constant of integration.

Proof. The proof of the Theorem 2.1 is resumed as follows: firstly, we apply the suitable functional transformation

$$z(x) = 1/[u(x)] \quad \Rightarrow \quad z'_x = -u'_x/u^2 \tag{4}$$

Without loss of generality, equation (4) removes the cubic nonlinearity extension term from equation (1), then, for $f_0(x) = 0$, we obtain the Abel's equation of the second kind

$$-[g_1(x) + g_0(x)u]u'_x = f_3(x) + f_2(x)u + f_1(x)u^2. \tag{5}$$

Multiplying both sides of equation (5) by an integrating factor $\mu_1 = \mu_1(x)$

$$-\mu_1(x)g_0(x)uu'_x - \mu_1(x)f_1(x)u^2 - \mu_1(x)g_1(x)u'_x = \mu_1(x)f_3(x) + \mu_1(x)f_2(x)u$$

and taking into account that $\mu'_1(x)g_0(x) = 2\mu_1(x)f_1(x)$, we have

$$-\frac{1}{2}g_0(x)(\mu_1(x)u^2)'_x - \mu_1(x)g_1(x)u'_x - \mu_1(x)f_2(x)u = \mu_1(x)f_3(x) \tag{6}$$

Multiplying both sides of equation (6) by an integrating factor $\mu_2 = \mu_2(x)$

$$-\frac{1}{2}\mu_2(x)g_0(x)(\mu_1(x)u^2)'_x - \mu_2(x)\mu_1(x)g_1(x)u'_x - \mu_2(x)\mu_1(x)f_2(x)u = \mu_2(x)\mu_1(x)f_3(x)$$

and taking into account that $\mu'_2(x)g_1(x) = \mu_2(x)f_2(x)$, we get

$$-\mu_2(x)g_0(x)(\mu_1(x)u^2)'_x - 2\mu_1(x)g_1(x)(\mu_2(x)u)'_x = 2\mu_2(x)\mu_1(x)f_3(x). \tag{7}$$

Dividing both sides of equation (7) by $-\mu_2(x)g_0(x)$ and from equation (2), we obtain

$$(\mu_1(x)u^2)'_x + \gamma(\mu_2(x)u)'_x = -2\frac{\mu_1(x)f_3(x)}{g_0(x)}. \tag{8}$$

After integrating equation (8), we use the relation (4) for returning to the original dependent variable z, so we obtain equation (3). This completes the proof of the Theorem. It is evident, therefore, that the proof is algorithmic because this demonstration is programmable. □

References

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