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Asymptotic Behaviour of a Viscoelastic Transmission Problem with a Tip Load

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Abstract. We consider a transmission problem for a string composed by two components: one of them is a viscoelastic material (with viscoelasticity of memory type), and the other is an elastic material (without dissipation effective over this component). Additionally, we consider that in one end is attached a tip load. The main result is that the model is exponentially stable if and only if the memory effect is effective over the string. When there is no memory effect, then there is a lack of exponential stability, but the tip load produces a polynomial rate of decay. That is, the tip load is not strong enough to stabilize exponentially the system, but produces a polynomial rate of decay.

Keywords. Transmission problems, memory effect, lack of exponential stability, tip load, hybrid system

1 Introduction

We consider the transmission problem for the damped vibrations of a string, whose left end is rigidly attached and in the other end has an attached hollow-tip body that contains granular material (Fig.1). The string is composed by two components: one of them is a viscoelastic material (with viscoelasticity of memory type) and the other is an elastic material (without dissipation effective over this component).

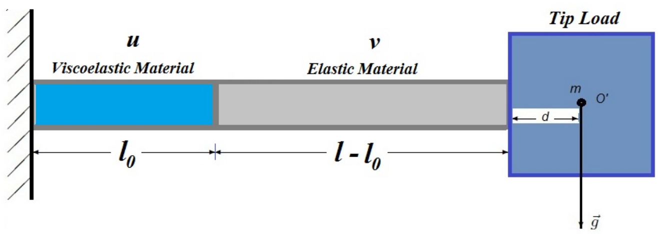

Figure 1: String with Tip Load.

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More precisely, let us denote by U the displacement of the string. That is

$$U(x) = \begin{cases} u(x), & x \in]0, l_0[\\ v(x), & x \in]l_0, l[\end{cases}$$

where l is the length of the string and l_0 is the transmission point. The model that we consider in this paper is written as follows.

$$\rho_1 u_{tt} - \alpha_1 u_{xx} + \int_0^t g(t-s) u_{xx}(\cdot, s) ds = 0 \quad \text{in} \quad]0, l_0[\times]0, +\infty[\tag{1}$$

$$\rho_2 v_{tt} - \alpha_2 v_{xx} = 0 \quad \text{in} \quad]l_0, l[\times]0, +\infty[. \tag{2}$$

Here, $g : [0, +\infty) \to \mathbb{R}$ be the relaxation function, and α_1, α_2, ρ_1, ρ_2 are positive constants that reflect physical properties of the string. The boundary conditions are given by

$$u(0, t) = 0, \qquad v(l, t) = w(t), \qquad \forall \ t \geq 0, \tag{3}$$

and the transmission conditions are given by

$$u(l_0, t) = v(l_0, t), \qquad \alpha_1 u_x(l_0, t) - \int_0^t g(t-s) u_x(l_0, s) ds = \alpha_2 v_x(l_0, t), \qquad \forall \ t \geq 0. \tag{4}$$

We turn to model the motion of the right end with the attached tip body. We assume that the container is rigidly attached to the end $x = l$, and that the container and its contents have mass m and a center of mass O' located at distance d from the end of the string. We assume that the damping effect of the internal granular material can be represented by damping coefficient γ_1, whose precise contributions are described by $\rho_3 w_{tt} + \gamma_1 w_t + \gamma_2 w$. Here, the first term is the contribution of the inertia of the container, and the second term represents the damping that the granular material provides, which is assumed to be proportional to the velocity, and so γ_1 is the damping coefficient. Thus, the force balance at the end $x = l$ is

$$\rho_3 w_{tt} + \gamma_1 w_t + \gamma_2 w + \alpha_2 v_x(l, .) = 0 \qquad \text{in} \qquad]0, +\infty[, \tag{5}$$

where the parameters γ_1 and γ_2 are non-negative constants. Finally, the initial data are given by

$$u(0) = u_0, u_t(0) = u_1 \text{ in }]0, l_0[, \quad v(0) = v_0, v_t(0) = v_1 \text{ in }]l_0, l[, \quad w(0) = w_0 \in \mathbb{C}, \ w_t(0) = w_1 \in \mathbb{C}. \tag{6}$$

Here, we assume the following hypotheses on the relaxation function g

$$g(t) \geq 0, \quad \forall t \geq 0, \quad \text{and} \quad g > 0 \text{ almost everywhere in } [0, +\infty[; \tag{7}$$

$$\exists \ k_1, k_2 > 0 : \ -k_1 g(t) \leq g'(t) \leq -k_2 g(t), \quad \forall t \geq 0; \tag{8}$$

$$0 < \alpha := \alpha_1 - \int_0^\infty g(s) ds. \tag{9}$$

Concerning models of motion with the attached tip body, Andrews and Shillor [1] establish the existence and uniqueness of the model and showed the exponential energy decay of the solution provided and extra damping term is present. See also the work of Feireisl and O'Dowd [7] where is showed, for an hybrid system composed of a cable with masses at both end, the strong stability for a nonlinear and nonmonotone feedback law applied at one end.

2 Existence and Uniqueness of Solutions

To use the semigroup approach we need to rewrite the problem as an autonomous system. For this reason we introduce the *history problem*, obtained by replacing the equation (1) by the following history equation

$$\rho_1 u_{tt} - \alpha_1 u_{xx} + \int_{-\infty}^{t} g(t-s) u_{xx}(.,s) ds = 0 \qquad \text{in} \qquad]0,l_0[\times]0,+\infty[.$$

Following the ideas of Dafermos [3], [4] and Fabrizio [6], we introduce the notation $\eta(x,t,s) := u(x,t) - u(x,t-s)$, with $s \in [0,+\infty)$; whence we consider the system

$$\rho_1 u_{tt} - \alpha u_{xx} - \int_0^\infty g(s)\eta_{xx}(s) ds = 0 \quad \text{in} \quad]0,l_0[\times]0,+\infty[\tag{10}$$

$$\rho_2 v_{tt} - \alpha_2 v_{xx} = 0 \quad \text{in} \quad]l_0,l[\times]0,+\infty[\tag{11}$$

$$\eta_t + \eta_s - u_t = 0 \quad \text{in} \quad]0,l_0[\times]0,+\infty[\times]0,+\infty[. \tag{12}$$

with u, v and w, satisfying (5) and the initial conditions (6) and η verifying

$$\eta(x,0,s) = \eta_0(x,s) =: u_0(x) - u_0(x,-s), \qquad \forall\ (x,s) \in\ ]0,l_0[\times]0,+\infty[, \tag{13}$$

with boundary conditions are given by

$$\eta(x,t,0) = 0,\ \forall\ (x,t) \in]0,l_0[\times]0,+\infty[, \qquad \eta(0,t,s) = 0,\ \forall\ (t,s) \in]0,+\infty[\times]0,+\infty[. \tag{14}$$

The transmission conditions now are given by

$$u(l_0,t) = v(l_0,t), \qquad \alpha u_x(l_0,t) + \int_0^\infty g(s)\eta_x(l_0,t,s) ds = \alpha_2 v_x(l_0,t), \qquad \forall\ \ t \geq 0. \tag{15}$$

We define the total energy of the system as

$$E(t) = \frac{1}{2}\left\{\int_0^{l_0}\left[\rho_1|u_t|^2 + \alpha|u_x|^2 + \int_0^\infty g(s)|\eta_x(s)|^2 ds\right]dx + \int_{l_0}^{l}\left[\rho_2|v_t|^2 + \alpha_2|v_x|^2\right]dx + \rho_3|w_t|^2 + \gamma_2|w|^2\right\}$$

Let us introduce the following spaces:

$$\mathbb{H}^m := H^m(0,l_0) \times H^m(l_0,l), \qquad m \in \mathbb{N};$$

$$\mathbb{H}^m_* := \{(u,v) \in \mathbb{H}^m;\ u(0) = 0,\ u(l_0) = v(l_0)\}, \quad m \in \mathbb{N};$$

$$\mathbb{L}^2 := L^2(0,l_0) \times L^2(l_0,l);$$

$$H^m_*(0,l_0) := \{f \in H^m(0,l_0);\ f(0) = 0\}, \quad m \in \mathbb{N};$$

$$L^2_g := \left\{\varphi : \mathbb{R}^+ \to H^1_*(0,l_0);\quad \int_0^\infty g(s)\int_0^{l_0}|\varphi_x(s)|^2 dx ds < \infty\right\}.$$

We recall that L^2_g is a Hilbert space when endowed with the inner product given by

$$\langle \varphi, \psi \rangle_{L^2_g} = \int_0^\infty g(s) \int_0^{l_0} \varphi_x(s)\overline{\psi_x(s)} dx ds.$$

We consider the phase space $\mathcal{H} := \big\{(u,v,U,V,\eta,w,W)^T \in \mathbb{H}^1_* \times \mathbb{L}^2 \times L^2_g \times \mathbb{C}^2;\ \ v(l) = w\big\}$. Note that the space $\mathcal{H}$ is a Hilbert space with the norm

$$\|\mathcal{U}\|^2_{\mathcal{H}} = \alpha\|u_x\|^2_{L^2(0,l_0)} + \alpha_2\|v_x\|^2_{L^2(0,l_0)} + \rho_1\|U\|^2_{L^2(0,l_0)} + \rho_2\|V\|^2_{L^2(0,l_0)} + \|\eta\|^2_{L^2_g} + \gamma_2|w|^2 + \rho_3|W|^2 \quad (16)$$

where $\mathcal{U} = (u,v,U,V,\eta,w,W)^T \in \mathcal{H}$.

Let us introduce the linear unbounded operator $\mathcal{A}$ in $\mathcal{H}$ as follows:

$$\mathcal{A}\,\mathcal{U} = \begin{pmatrix} U \\ V \\ \dfrac{\alpha}{\rho_1}u_{xx} + \dfrac{1}{\rho_1}\displaystyle\int_0^\infty g(s)\eta_{xx}(s)ds \\ \dfrac{\alpha_2}{\rho_2}v_{xx} \\ U - \eta_s \\ W \\ -\dfrac{\gamma_1}{\rho_3}W - \dfrac{\gamma_2}{\rho_3}w - \dfrac{\alpha_2}{\rho_3}v_x(l) \end{pmatrix}$$

with domain

$$D(\mathcal{A}) = \left\{ \begin{array}{c} \mathcal{U} = (u,v,U,V,\eta,w,W)^T \in \mathcal{H};\ \left(\alpha u + \displaystyle\int_0^\infty g(s)\eta(s)ds,\ v\right) \in \mathbb{H}^2,\ (U,V) \in \mathbb{H}^1_*, \\ V(l) = W, \qquad \eta|_{s=0} = 0, \qquad \eta_s \in L^2_g, \qquad \alpha u_x(l_0) + \displaystyle\int_0^\infty g(s)\eta_x(l_0,s)ds = \alpha_2 v_x(l_0) \end{array} \right\}.$$

Using the hypotheses on g, a direct computation yields

$$\operatorname{Re}\langle \mathcal{A}\,\mathcal{U}, \mathcal{U}\rangle = -\gamma_1|W|^2 + \frac{1}{2}\int_0^{l_0}\!\!\int_0^\infty g'(s)|\eta_x(s)|^2 ds dx \ \le\ 0,$$

which means that $\mathcal{A}$ is a dissipative operator. The system (10)-(15) is equivalent to

$$\mathcal{U}_t = \mathcal{A}\,\mathcal{U}, \qquad \mathcal{U}(0) = \mathcal{U}_0; \qquad (17)$$

where $\mathcal{U}(t) = (u(t),v(t),U(t),V(t),\eta(t),w(t),W(t))^T$ and $\mathcal{U}_0 = (u_0,v_0,u_1,v_1,\eta_0,w_0,w_1)^T$.

Under this conditions, we have

Teorema 2.1. *The operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions $(S(t))_{t\ge 0}$ on $\mathcal{H}$. Thus, for any initial data $\mathcal{U}_0 \in \mathcal{H}$, the problem (17) has a unique weak (mild) solution*

$$\mathcal{U} \in \mathcal{C}^0([0,\infty[\,,\mathcal{H}).$$

Moreover, if $\mathcal{U}_0 \in D(\mathcal{A})$, then $\mathcal{U}$ is a strong solution of (17), that is

$$\mathcal{U} \in \mathcal{C}^1([0,\infty[\,,\mathcal{H}) \cap \mathcal{C}^0([0,\infty[\,,D(\mathcal{A})).$$

Proof. It easy to see that $D(\mathcal{A})$ is dense in $\mathcal{H}$; and, since $\mathcal{A}$ is a dissipative operator, it is enought to show that $0 \in \rho(\mathcal{A})$. For this, it is proved that for $F = (f^1, f^2, \cdots, f^7)^T \in \mathcal{H}$, there exists only one $\mathcal{U} = (u,v,U,V,\eta,w,W)^T \in D(\mathcal{A})$ such that $\mathcal{A}\,\mathcal{U} = F$. $\square$

3 Exponential stability

Teorema 3.1. *Let us suppose that (7)-(9) hold. Then the semigroup $e^{\mathcal{A}t}$ is exponentially stable.*

Proof. The main tool we use is Prüss's results [9], which is summarized in the following theorem.

Teorema 3.2. *Let $(\mathcal{S}(t))_{t\geq 0}$ be a C_0-semigroup on a Hilbert space $\mathcal{H}$ generated by $\mathcal{A}$. Then the semigroup is exponentially stable if and only if*

$$i\mathbb{R} \subset \rho(\mathcal{A}), \quad \textit{and} \quad \|(i\,\lambda\, I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C, \quad \forall\,\lambda \in \mathbb{R}.$$

Since the resolvent operator is holomorphic, it is enough to prove that $\|(i\,\lambda\, I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C$ for $|\lambda|$ large enough. This is established by the following inequalities:

$$\textbf{[I 1]:}\ \ \gamma_1|W|^2 + \int_0^{l_0}\int_0^\infty g(s)|\eta_x(s)|^2 ds dx \leq C\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

For any $\epsilon > 0$ sufficiently small and $|\lambda|$ large enough, hold:

$$\textbf{[I 2]:}\quad \int_0^{l_0} |U|^2 + |u_x|^2 dx \leq C_\epsilon\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C_\epsilon\|F\|^2_{\mathcal{H}} + \epsilon\left|\alpha u_x(l_0) + \int_0^\infty g(s)\eta_x(l_0,s)ds\right|^2,$$

$$\textbf{[I 3]:}\quad \rho_2|W|^2 \leq C\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\int_0^{l_0}\alpha|v_x|^2 dx,$$

$$\textbf{[I 4]:}\quad \int_{l_0}^{l}\left[\alpha_2|v_x|^2 + \rho_2|V|^2\right]dx \leq C\|\mathcal{U}\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|^2_{\mathcal{H}}.$$

□

4 The Lack of Exponential Stability

In this section, it is proved that when there is no memory effect, i.e., when $g = 0$ the system is not exponentially stable. Here α_1, α_2, ρ_1, ρ_2, ρ_3, γ_2 are as before, and γ_1, now, is a positive constant. Moreover, for this problem, we consider the phase space

$$\breve{\mathcal{H}} = \left\{\mathcal{U} = (u, v, U, V, w, W)^T \in \mathbb{H}^1_* \times \mathbb{L}^2 \times \mathbb{C}^2;\ \ v(l) = w\right\}.$$

Let us denote by $\mathcal{B}$ the unbounded operator of $\breve{\mathcal{H}}$ given by

$$\mathcal{B}\,\mathcal{U} = \left(U, V, \frac{\alpha_1}{\rho_1}u_{xx}, \frac{\alpha_2}{\rho_2}v_{xx}, W, -\frac{\gamma_1}{\rho_3}W - \frac{\gamma_2}{\rho_3}w - \frac{\alpha_2}{\rho_3}v_x(l)\right)^T$$

with domain $D(\mathcal{B}) = \left\{\mathcal{U} = (u, v, U, V, w, W)^T \in (\mathbb{H}^1_* \cap \mathbb{H}^2)\times\mathbb{H}^1_*\times\mathbb{C}^2; V(l) = W, \alpha_1 u_x(l_0) = \alpha_2 v_x(l_0)\right\}$. It is not difficult to see that the operator $\mathcal{B}$ is the infinitesimal generator of a C_0-semigroup of contractions over $\breve{\mathcal{H}}$, which we will denote by $T(t)$. This shows that the problem without memory effect is well-posed.

The main tool we use to prove that the system is not exponentially is the Weyl's theorem about the invariance of the essential spectral radius by compact perturbations. To do that, let us consider the following *conservative system*

$$\rho_1\tilde{u}_{tt} - \alpha_1\tilde{u}_{xx} = 0 \quad \text{in} \quad]0, l_0[\ \times\ ]0, +\infty[\tag{18}$$

$$\rho_2\tilde{v}_{tt} - \alpha_2\tilde{v}_{xx} = 0 \quad \text{in} \quad]l_0, l[\ \times\ ]0, +\infty[\tag{19}$$

$$\rho_3\tilde{w}_{tt} + \gamma_2\tilde{w} + \alpha_2\tilde{v}_x(l) = 0 \quad \text{in} \quad]0, +\infty[\tag{20}$$

verifying the same boundary and transmission conditions and with the same initial data, where α_1, α_2, ρ_1, ρ_2, ρ_3 and γ_2 are as before. That is, with boundary conditions:

$$\tilde{u}(0) = 0, \quad \tilde{v}(l) = \tilde{w} \quad \text{in} \quad]0, +\infty[\tag{21}$$

and transmission conditions:

$$\tilde{u}(l_0) = \tilde{v}(l_0), \quad \alpha_1 \tilde{u}_x(l_0) = \alpha_2 \tilde{v}_x(l_0) \quad \text{in} \quad]0, +\infty[\tag{22}$$

and initial data

$$(\tilde{u}(0), \tilde{v}(0), \tilde{u}_t(0), \tilde{v}_t(0), \tilde{w}(0), \tilde{w}_t(0)) = (u_0, v_0, u_1, v_1, w_0, w_1) \in \breve{\mathcal{H}}. \tag{23}$$

The total energy associed with this system is

$$\tilde{E}(t) = \frac{1}{2}\left[\int_0^{l_0} [\rho_1|\tilde{u}_t|^2 + \alpha_1|\tilde{u}_x|^2]\,dx + \int_{l_0}^{l} [\rho_2|\tilde{v}_t|^2 + \alpha_2|\tilde{v}_x|^2]\,dx + \rho_3|\tilde{w}_t|^2 + \gamma_2|\tilde{w}|^2\right],$$

and it is not difficult to see that $\dfrac{d}{dt}\tilde{E}(t) = 0$. Therefore the system is conservative and there is no decay. Now we are in conditions to show the main result of this section.

Teorema 4.1. *The semigroup $T(t)$ associated to system without memory effect is not exponentially stable.*

Proof. The main idea of the proof is to show that the semigroup $T(t)$ have the same essential spectral radius of the semigroup associated to conservative system (18)-(23), that we denote as $T_0(t)$. Here, we use the follows Weyl's Theorem (see [8], Theorem 5.35, p. 244)

Teorema 4.2 (Weyl)**.** *Let S and K two continuous operator over a Banach space X. If $S - K$ is a compact operator, then S and K have the same essential spectrum radius.*

The difference $T(t) - T_0(t)$ is a compact operator, from which we obtain $\omega_{ess}(T) = \omega_{ess}(T_0)$. But since $T_0(t)$ is unitary, then $\omega_{ess}(T_0) = 0$. Denoting by $\omega(T)$ and $\omega_\sigma(\mathcal{B})$ the type of semigroup $T(t)$ and the spectral upper bound of spectrum $\sigma(\mathcal{B})$ respectively, we have (see [5], Corollary 2.11) that $\omega(T) = \max\{\omega_\sigma(\mathcal{B}), \omega_{ess}(T)\} = 0$. This imply that $T(t)$ is not exponentially stable. $\square$

5 Polynomial decay

Here is shown that the solutions of the system without memory effect decays polynomially to zero as $t^{-1/2}$. This means that the dissipation given by tip load produces a polynomial rate of decay. To show this, we use the Borichev and Tomilov's Theorem (see [2]):

Teorema 5.1. *Let $S(t)$ be a bounded C_0-semigroup on a Hilbert space $\mathcal{H}$ with generator $\mathcal{A}$ such that $i\mathbb{R} \subset \rho(\mathcal{A})$. Then*

$$\frac{1}{|\lambda|^\beta}\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \le C, \quad \forall\, \lambda \in \mathbb{R} \quad \Longleftrightarrow \quad \|S(t)\mathcal{A}^{-1}\|_{D(\mathcal{A})} \le \frac{C}{t^{1/\beta}}, \quad \forall\, t > 0.$$

Our starting point is to study the solution of the resolvent equation $i\lambda\mathcal{U} - \mathcal{B}\mathcal{U} = F$. From this equation, written in terms of its components, we obtain the following inequalities, for $|\lambda|$ large enough:

$$\textbf{[I 5]:} \quad \gamma_1|W|^2 \le C\|\mathcal{U}\|\|F\|;$$

$$\textbf{[I 6]:} \quad \int_{l_0}^{l} [\rho_2|V|^2 + \alpha_2|v_x|^2]dx \le C|\lambda|^2\|\mathcal{U}\|\|F\| + C\|F\|^2;$$

$$\textbf{[I 7]:} \quad \int_0^{l_0} [\rho_1|U|^2 + \alpha_1|u_x|^2]dx \le C|\lambda|^2\|\mathcal{U}\|\|F\| + C\|F\|^2.$$

Teorema 5.2. *The semigroup $T(t)$ associated to system without memory effect decays polynomially as $t^{-1/2}$ as $t \to \infty$. Moreover, if $\mathcal{U}_0 \in D(\mathcal{B}^k)$, then*

$$\|T(t)\,\mathcal{U}_0\|_{\breve{\mathcal{H}}} \leq \frac{C_k}{t^{k/2}}\|\mathcal{U}_0\|_{D(B^k)}.$$

Proof. From the inequalities above, for $|\lambda|$ large enough, follows that $\|\mathcal{U}\|^2_{\breve{\mathcal{H}}} \leq C|\lambda|^4\|F\|^2_{\breve{\mathcal{H}}}$. Then, for $|\lambda|$ large enough, $\|(i\lambda I - \mathcal{B})^{-1}F\|_{\breve{\mathcal{H}}} = \|\mathcal{U}\|_{\breve{\mathcal{H}}} \leq C|\lambda|^2\|F\|_{\breve{\mathcal{H}}}$. Therefore, from Borichev and Tomilov's Theorem our conclusion follows. □

6 Conclusões

The main result of this paper was to show that the system (1)–(6) is exponentially stable if and only if the memory effect is effective over the viscoelastic part of the material. This means that the dissipative properties given by the tip load is not enough to produce exponencial rate of decay when the memory effect is not effective. Finally, when $g = 0$, we prove that the system is not exponentially stable but the dissipation given by the tip load produce polynomial stability.

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