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The Contradiction Argument for the Brouwer Conjecture.

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Resumo. In this work we develop results to contribute to the study of the Brouwer conjecture through an argument of contradiction. Specifically, if the Brouwer conjecture is not valid for a graph and this graph respects some conditions, we show that the conjecture is also not valid for the graphs obtained by deleting an edge or vertex. In this way, we can recursively delete certain vertices and edges of the original graph until the resulting graph is in a family for which the conjecture is proven, a contradiction.

Palavras-chave. Brouwer Conjecture, Laplacian Eigenvalues, Spectral Graph Theory.

1 Introduction

Let $G = (V(G), E(G))$ be a simple graph with order $n = |V(G)|$ and size $e(G)$, where $V(G) = \{v_1, ..., v_n\}$ is the vertex set and $E(G)$ is the edge set.

The degree $d_i(G) = d_{v_i}(G)$ of a vertex v_i is the number of its neighbors, that is, it is the number of edges incident in v_i. The degree sequence of a graph G is given by $d_1(G) \geq \cdots \geq d_n(G)$, where $d_i(G)$ is the i-th largest degree of G. We drop G from the notation when there is no danger of confusion.

The adjacency matrix of G is an $n \times n$ matrix $A(G) = [a_{ij}]$, where $a_{ij} = 1$ if v_i and v_j are adjacent and $a_{ij} = 0$ otherwise. The Laplacian matrix is the matrix $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix with the degree sequence of G on its main diagonal.

It is well-known that $L(G)$ is a positive semidefinite matrix and so its eigenvalues are nonnegative real numbers. The eigenvalues of $L(G)$ are called the Laplacian eigenvalues of G and are denoted by $\mu_1(G) \geq \cdots \geq \mu_n(G) = 0$.

Grone and Merris [7] investigated the relation between the sum of the largest Laplacian eigenvalues and the sum of the largest degrees of a graph and stated the following conjecture.

Grone and Merris Conjecture: Let G be a graph with n vertices. Then

$$\sum_{i=1}^{k} \mu_i(G) \leq \sum_{i=1}^{k} \#\{l | d_l \geq i\}, \text{ for } k = 1, ..., n. \tag{1}$$

This conjecture was recently proved by Hua Bai [2].

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As a variation on the Grone–Merris conjecture, Brouwer [4] stated a conjecture that relates the sum of the k largest Laplacian eigenvalues with the number of edges of the graph plus a combinatorial factor that depends on the value k chosen.

Brouwer Conjecture: Let G be a graph with n vertices. Then

$$\sum_{i=1}^{k} \mu_i(G) \leq e(G) + \binom{k+1}{2}, \text{ for } k = 1, ..., n. \tag{2}$$

This conjecture is known to be true for:
(i) $k = n$ and $k = n - 1$ [14];
(ii) $k = 1$ by the well-known inequality $\mu_1(G) \leq n$ [12];
(iii) $k = 2$ [8];
(iv) trees [6];
(v) unicyclic and bicyclic graphs [5, 14];
(vi) regular graphs [3, 11];
(vii) split graphs [3, 11];
(viii) cographs [3, 11];
(ix) graphs with at most 10 vertices [3, 11].

The Brouwer conjecture is directly related to the study of Laplacian graph energy. According to [10], the validity of the conjecture ensures that for every graph G, there exists a Threshold graph T with the same order and size, so that the Laplacian energy of T is greater than or equal to the Laplacian energy of G. Therefore, for a fixed number of vertices and edges, the maximum Laplacian energy is reached in a threshold graph.

In this work, we present some tools for the study of Brouwer's conjecture through the contradiction argument. We show that if the Brouwer conjecture is not valid for a graph that respects some conditions associated with upper bounds for its largest Laplacian eigenvalue, then the conjecture is also not valid for the graph resulting from the deletion of a edge. A similar result is presented for the graph resulting from the deletion of a vertex.

The remainder of this paper is organized as follows. In the next section we present the notations and preliminaries used. In section 3 we discuss the contradiction argument for the Brouwer conjecture and develop our main results associated with the deletion of vertices and edges. Finally, in the last section we show some applications, discuss our conclusions and present possible future works.

2 Preliminaries

Let $G = (V(G), E(G))$ be a simple graph with order n. The graph obtained from G by deleting the edge $e \in E(G)$, denoted by $G-e$, is the graph with $V(G-e) = V(G)$ and $E(G-e) = E(G)-\{e\}$. Van den Huevel [13] shows that the Laplacian eigenvalues of the graph G and graph $G - e$ are related by the following interlacing theorem.

Theorem 2.1. *Let G be a graph with n vertices and let $G - e$ be a subgraph of G obtained by deleting the edge $e \in E(G)$. Then*

$$\mu_1(G) \geq \mu_1(G - e) \geq \mu_2(G) \geq \cdots \geq \mu_{n-1}(G - e) \geq \mu_n(G) \geq \mu_n(G - e). \tag{3}$$

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The graph obtained from G deleting the vertex $v \in V(G)$, denoted by $G - v$, is the graph with $V(G - v) = V(G) - \{v\}$ and $E(G - v) = E(G) - \{uv | uv \in E(G)\}$. We enunciate the interlacing theorem for the Laplacian eigenvalues of G and $G - v$ below [13].

Theorem 2.2. *Let G be a graph with n vertices and let $G - v$ be a subgraph of G obtained by deleting the vertex $v \in V(G)$. Then*

$$\mu_i(G) \geq \mu_i(G - e) \geq \mu_{i+d_v}(G), \ for \ i = 1, ..., n, \tag{4}$$

where $\mu_i(G) = 0$ for $i \geq n + 1$.

The vertex and edge deletion operations, and their associated interlacing theorems, will be used in the development of the main results presented in the next section.

The Laplacian eigenvalues have applications in several fields, such as randomized algorithms, combinatorial optimization problems and machine learning. In the spectral graph theory, a fundamental problem is the search for bounds for Laplacian eigenvalues. Anderson and Morley [1] presented the following upper bound for the largest Laplacian eigenvalue.

Theorem 2.3. *Let G be a simple graph. Then*

$$\mu_1(G) \leq \max\{d_u + d_r | v_u v_r \in E(G)\}.$$

Zhang [15] surveyed the known results for Laplacian eigenvalues and shows that there are several bounds for the largest Laplacian eigenvalue that are based on different graph parameters. We will explore this fact to develop applications of the main results in the last section.

3   The Contradiction Argument for the Brouwer Conjecture

It is well known that the Brouwer conjecture is valid for the different families of graphs presented in the Section 1. We intend to use these facts to study a proof for Brouwer's conjecture through the contradiction argument.

Precisely, if the Brouwer conjecture is not valid for a graph and this graph respects some conditions, we show that the conjecture is also not valid for the graphs obtained by deleting an edge or vertex. In this way, we can recursively delete certain vertices and edges of the original graph until the resulting graph is in a family for which the conjecture is proven, a contradiction.

In the following, we present the result associated with the deletion of an edge.

Theorem 3.1. *If the j-th inequality of the Brouwer conjecture is not valid for a graph G with n vertices and $\mu_1(G) \leq j + 1$. Then the Brouwer conjecture is not valid for $G - e$, where $e \in E(G)$.*

Proof. Suppose that the j-th inequality of the Brouwer conjecture is not valid for a graph G. Specifically,

$$\sum_{i=1}^{j} \mu_i(G) > e(G) + \binom{j+1}{2} \tag{5}$$

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As the Brouwer conjecture is proved for $k = 1, 2, n-1$ and n, we assume that $3 \leq j \leq n-2$. Summing the $j-1$ largest Laplacian eigenvalues of $G-e$ and using the inequalities of **Theorem 2.1**, we have

$$\sum_{i=1}^{j-1} \mu_i(G-e) \geq \sum_{i=2}^{j} \mu_i(G) = \sum_{i=1}^{j} \mu_i(G) - \mu_1(G). \tag{6}$$

From inequality (5) follows

$$\sum_{i=1}^{j-1} \mu_i(G-e) \ > \ e(G) + \binom{j+1}{2} - \mu_1(G)$$

$$= \ e(G-e) + \binom{j}{2} + [j+1-\mu_1(G)]. \tag{7}$$

From the hypothesis we have $j + 1 - \mu_1(G) \geq 0$, and we conclude that the Brouwer conjecture is not valid of $G - e$ for the sum of the eigenvalues up to $j-1$.

$\square$

Below we present the result associated with the deletion of a vertex with any degree.

Theorem 3.2. *Let G be a graph with n vertices and v a vertex with $d_v \leq n-5$. If the $(j+d_v)$-th inequality of the Brouwer conjecture is not valid for G and*

$$\mu_1(G) + \cdots + \mu_{d_v}(G) \leq \frac{d_v(2j + d_v + 3)}{2}.$$

Then the Brouwer conjecture is not valid for $G - v$.

Proof. Suppose that the $(j + d_v)$-th inequality of the Brouwer conjecture is not valid for a graph G. Specifically,

$$\sum_{i=1}^{j+d_v} \mu_i(G) > e(G) + \binom{j + d_v + 1}{2} \tag{8}$$

As the Brouwer conjecture is proved for $k = 1, 2, n-1$ and n, we assume that $2 \leq j \leq n-d_v-2$. Summing the j largest Laplacian eigenvalues of $G-v$ and using the inequalities of **Theorem 2.2**, we have

$$\sum_{i=1}^{j} \mu_i(G-v) \geq \sum_{i=d_v+1}^{j+d_v} \mu_i(G) = \sum_{i=1}^{j+d_v} \mu_i(G) - [\mu_1(G) + \cdots + \mu_{d_v}(G)]. \tag{9}$$

From inequality (8) follows

$$\sum_{i=1}^{j} \mu_i(G-v) \ > \ e(G) + \binom{j + d_v + 1}{2} - [\mu_1(G) + \cdots + \mu_{d_v}(G)]$$

$$= \ e(G) + \frac{(j + d_v + 1)(j + d_v)}{2} - [\mu_1(G) + \cdots + \mu_{d_v}(G)]$$

$$= \ e(G) + \frac{j(j+1)}{2} + \frac{d_v(2j + d_v + 1)}{2} - [\mu_1(G) + \cdots + \mu_{d_v}(G)]$$

$$= \ e(G-v) + \binom{j+1}{2} + \left[\frac{d_v(2j + d_v + 3)}{2} - \mu_1(G) - \cdots - \mu_{d_v}(G) \right] \tag{10}$$

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From the hypothesis we have

$$\frac{d_v(2j + d_v + 3)}{2} - \mu_1(G) - \cdots - \mu_{d_v}(G) \geq 0,$$

and we conclude that the Brouwer conjecture is not valid of $G - v$ for the sum of the eigenvalues up to j.

$\square$

The deletion of a vertex with a large degree provides a condition with many parameters to be studied. Therefore, we present a corollary for the deletion of a vertex of degree one that provides the same condition as the edge deletion case.

Corollary 3.1. *Let G be a graph with n vertices and v a vertex with $d_v = 1$. If the j-th inequality of the Brouwer conjecture is not valid for G and $\mu_1(G) \leq j + 1$. Then the Brouwer conjecture is not valid for $G - v$.*

Proof. This proof is analogous to the previous result, using **Theorem 2.2** with $d_v = 1$.

$\square$

This corollary can be used to study the Brouwer conjecture for graphs that have leaves.

4 Conclusions and Applications

In this section, we will discuss some of the conclusions and applications of the results developed in section 3. First, we present some variations of the main results that can be constructed using upper bounds for the largest Laplacian eigenvalue. Finally, we discuss the study of the Brouwer conjecture using the results obtained and their applications in future works.

We can develop variations for the main results using different upper bounds for the Laplacian eigenvalues. Assuming that the Brouwer conjecture is not valid for a graph, we can study the conditions that the original graph must respect on different parameters so that the Brouwer conjecture is also not valid for the graphs obtained by deleting an edge or a vertex.

As a simple example, we use the upper bound for the largest Laplacian eigenvalue given by **Theorem 2.3** to develop the following result.

Theorem 4.1. *If the j-th inequality of the Brouwer conjecture is not valid for a graph G with n vertices and $\max\{d_u + d_r | v_u v_r \in E(G)\} \leq j + 1$. Then the Brouwer conjecture is not valid for $G - e$, where $e \in E(G)$.*

Proof. Suppose that the j-th inequality of the Brouwer conjecture is not valid for a graph G. Specifically,

$$\sum_{i=1}^{j} \mu_i(G) > e(G) + \binom{j+1}{2} \tag{11}$$

As the Brouwer conjecture is proved for $k = 1, 2, n - 1$ and n, we assume that $3 \leq j \leq n - 2$.

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Summing the $j-1$ largest Laplacian eigenvalues of $G-e$ and using the inequalities of **Theorem 2.1**, we have

$$
\begin{aligned}
\sum_{i=1}^{j-1} \mu_i(G - e) &\geq \sum_{i=2}^{j} \mu_i(G) \\
&= \sum_{i=1}^{j} \mu_i(G) + (\max\{d_u + d_r | v_u v_r \in E(G)\}) - (\max\{d_u + d_r | v_u v_r \in E(G)\}).
\end{aligned}
$$

Using the bound of **Theorem 2.3**, we have

$$
\sum_{i=1}^{j-1} \mu_i(G - e) \geq \sum_{i=1}^{j} \mu_i(G) - (\max\{d_u + d_r | v_u v_r \in E(G)\}). \tag{12}
$$

From inequality (11) follows

$$
\begin{aligned}
\sum_{i=1}^{j-1} \mu_i(G - e) &> e(G) + \binom{j+1}{2} - (\max\{d_u + d_r | v_u v_r \in E(G)\}). \\
&= e(G - e) + \binom{j}{2} + [j + 1 - (\max\{d_u + d_r | v_u v_r \in E(G)\})]. \tag{13}
\end{aligned}
$$

From the hypothesis we have $j + 1 - (\max\{d_u + d_r | v_u v_r \in E(G)\}). \geq 0$, and we conclude that the Brouwer conjecture is not valid of $G - e$ for the sum of the eigenvalues up to $j - 1$.

$\square$

This variation of **Theorem 3.1** associates the investigation of the Brouwer conjecture with the study of the maximum sum of the degrees of two adjacent vertices in the graph. Using other upper bounds for the largest Laplacian eigenvalue, we can obtain similar results to investigate the Brouwer conjecture by studying different graph parameters. Analogous results can also be developed in the case of deletion of a vertex.

The results obtained aim to investigate the validity of the Brouwer conjecture. Therefore, as a future work we intend to investigate in which graph families we can apply these results to ensure that they satisfy the conjecture.

In our computational experiments we have shown that the Brouwer conjecture is valid for some graphs through the process of vertex and edge deletion and arguing for contradiction. Our main future objective is to investigate the existence of a recursive process of deleting edges and vertices that can transform any graph into a graph that satisfies the Brouwer conjecture

References

[1] Anderson, W. N. and Morley, T. D. Eigenvalues of the Laplacian of a graph, *Linear and Multilinear Algebra*, 18:141-145, 1985. DOI: 10.1080/03081088508817681.

[2] Bai, H. The grone-merris conjecture. *Transactions of the American Mathematical Society*, 363:4463-4474, 2011. DOI: 10.2307/23032227.

[3] Berndsen, J. and Blokhuis, A. Three problems in algebraic combinatorics, Dissertação de Mestrado, Eindhoven University of Technology, 2012.

[4] Brouwer, A. E. and Haemers, W. H. *Spectra of graphs.* Springer, New York, 2012.

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[5] Du, Z. and Zhou, B. Upper bounds for the sum of Laplacian eigenvalues of graphs, *Linear Algebra and its Applications*, 436:3672-3683, 2012. DOI: 10.1016/j.laa.2012.01.007.

[6] Fritscher, E., Hoppen, C., Rocha, I. and Trevisan, V. On the sum of the Laplacian eigenvalues of a tree, *Linear Algebra and its Applications*, 435:371-399, 2011. DOI: 10.1016/j.laa.2011.01.036

[7] Grone, R. and Merris, R. The Laplacian spectrum of a graph II, *SIAM Journal on discrete mathematics*, 7:221-229, 1994. DOI: 10.1137/S0895480191222653.

[8] Haemers, W. H., Mohammadian, A. and Tayfeh-Rezaie, B. On the sum of Laplacian eigenvalues of graphs, *Linear Algebra and its Applications*, 432:2214-2221, 2010. DOI: 10.1016/j.laa.2009.03.038.

[9] Hall, F., Patel, K. and Stewart, M. Interlacing results on matrices associated with graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing*, 68:113-127, 2009. DOI: 10.1027/25632223433.

[10] Helmberg, C. and Trevisan, V. Threshold graphs of maximal Laplacian energy, *Discrete Mathematics*, 338:1075-1084, 2015. DOI: 10.1016/j.disc.2015.01.025.

[11] Mayank, O. Variants of the Grone–Merris conjecture, Dissertação de Mestrado, Eindhoven University of Technology, 2010.

[12] Merris, R. Laplacian matrices of graphs: a survey, *Linear Algebra and its Applications*, 197:143-176, 1994. DOI: 10.1016/0024-3795(94)90486-3.

[13] Van Den Heuvel, J. Hamilton cycles and eigenvalues of graphs, *Linear algebra and its applications*, 226:723-730, 1995. DOI: 10.1016/0024-3795(95)00254-O.

[14] Wang, S., Huang, Y. and Liu, B. On a conjecture for the sum of Laplacian eigenvalues, *Mathematical and Computer Modelling*, 56:60-68, 2012. DOI: 10.1016/j.mcm.2011.12.047.

[15] Zhang, X. D. The Laplacian eigenvalues of graphs: a survey, *arXiv preprint arXiv:1111.2897*, 2011.