Proceeding Series of the Brazilian Society of Computational and Applied Mathematics, v. 8, n. 1, 2021.

Trabalho apresentado no XL CNMAC, Evento Virtual - Co-organizado pela Universidade do Mato Grosso do Sul (UFMS).

Graph Invertibility based on Drazin Inverse.

André Luis Andrejew Ferreira[1]
UFPel, Pelotas, RS
Débora Marília Hauenstein[2]
UFPel, Pelotas, RS
Guilherme Porto[3]
IFFar, São Borja, RS

1 Introduction

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, ..., v_n\}$ and edge set $E(G)$. In this work, graphs may contain loops (edges that connects a vertex to itself) but no multiple edges.

A weighted graph is a graph in which a number (the weight) is assigned to each edge. Let (G, w) be a weighted graph, where $w : E(G) \to \mathbb{R} \setminus \{0\}$ is the function that assigns weights to the edges. The weighted adjacency matrix of (G, w) is $A(G, w) = [a_{ij}]$, where $a_{ij} = w(v_iv_j)$ if $v_iv_j \in E(G)$ and $a_{ij} = 0$ otherwise. If the weight of each edge is equal to one then $A(G, w)$ is the standard adjacency matrix of the graph G, denoted by $A(G)$.

Since there are many compositing operations on graphs, it is natural to question whether or not there is a notion of the inverse of a graph G with the property that $(G^{-1})^{-1}$ be isomorphic to G. In this work, we investigate this question and present an original approach to graph inversion based on the Drazin inverse.

The inversion of graphs was studied by several researchers and motivated the development of different notions, definitions and applications. Graph inverses have spectral graph theory interests and can be applied to bound median eigenvalues of graphs. If a graph has its spectra split about the origin, i.e., half of eigenvalues are positive and half of them are negative, then its median eigenvalues can be bounded by estimating the largest and smallest eigenvalues of its inverse [5].

Graph inversion provides a combinatorial interpretation of Motzkin numbers in terms of unicyclic graphs and has several applications in geometry, combinatorics and number theory [3]. The n-th Motzkin number is the number of different ways of drawing non-intersecting chords between n points on a circle (not necessarily touching every point by a chord).

In the next section, we review the definitions for graph inversion present in the literature. In the last section, as an original contribution, we present another notion for the inversibility of graphs based on the Drazin inverse matrix of the adjacency matrix and state some preliminary results.

2 Notions of Graph Invertibility

We summarize the definitions proposed in the works of Godsil [2], McLeman and McNicholas [3], and Yang and Ye [4] to establish a more comprehensive notion of the inverse of a graph.

Godsil defined an inverse of a bipartite graph G with a unique perfect matching to be a graph G^{-1} with adjacency matrix diagonally similar to the inverse of adjacency matrix of G, i.e., the

[1] andrejew.ferreira@gmail.com.
[2] debora.hauenstein@ufpel.edu.br.
[3] guilherme.porto@iffarroupilha.edu.br

adjacency matrix of G^{-1} is $\mathbb{D}A(G)^{-1}\mathbb{D}$ for some diagonal matrix $\mathbb{D}$ with entries 1 or -1 on its diagonal. McLeman and McNicholas extended Godsil's definition to all graphs (multigraphs and weighted graphs). Yang and Ye provides a complete characterization of invertible bipartite graphs with a unique perfect matching.

Each of these works presented its results with necessary and sufficient conditions to characterize the graphs that are invertible and to determine their inverses. Moreover, they discuss the difficulties and limitations arising from the definition.

3 The Graph Inversion based on Drazin's Inverse

The formulation of different notions for graph inversion seeks definitions that are more efficient to characterize graphs that are invertible, generalizing the previous approaches. In this sense, note that there are graphs whose adjacency matrix is not invertible, in addition, the inverse of the adjacency matrix of a graph is not necessarily an adjacency matrix of another graph. We deal with this question in our notion for inversion of graphs using the Drazin inverse.

The Drazin inverse matrix is a kind of generalized inverse of a matrix that has applications in systems of linear differential equations, Markov chain theory and population growth models. The Drazin inverse will only be defined for square matrices [1].

Let M be a square matrix. The smallest non-negative integer k such that $rank(M^{k+1}) = rank(M^k)$, is called the index of M and is denoted by $Ind(M)$. The Drazin inverse of M with $Ind(M) = k$ is the unique matrix M^D which satisfies the following properties:

$$M^D M M^D = M^D \qquad M M^D = M^D M \qquad M^{k+1} M^D = M^k$$

If M is a nonnegative symmetric matrix then M^D is a nonnegative symmetric matrix. Therefore, the Drazin inverse of the adjacency matrix of a graph represents the weighted adjacency matrix of a weighted graph. **We define G as an invertible graph if $A(G) = (A(G)^D)^D$ and its inverse is the weighted graph $G^{-1} = (G^{-1}, w)$ with weighted adjacency matrix given by $A(G^{-1}, w) = A(G)^D$.**

It is well known that if M is invertible with inverse M^{-1}, then $M^D = M^{-1}$. As a preliminary result, we can use the Drazin inverse to generalize the notions of graph inversion presented in the previous section. The main preliminary result obtained is the partial characterization of the invertible unicyclic graphs. The objective of this study is to state the necessary and sufficient conditions for a unicyclic graph to be invertible and, consequently, to provide a complete characterization of the invertible unicyclic graphs.

References

[1] Campbell, S. L. and Meyer, C. D. *Generalized inverses of linear transformations.* Society for industrial and applied Mathematics, 2009.

[2] Godsil, C. D. Inverses of trees, *Combinatorica*, 5:33-39, 1985. DOI:10.1007/BF02579440.

[3] McLeman, C. and McNicholas, E. Graph Invertibility. *Graphs and Combinatorics*, 30:977-1002, 2014. DOI:10.1007/s00373-013-1319-7.

[4] Yang, Y. and Ye, D. Inverses of bipartite graphs. *Combinatorica*, 38:1251-1263, 2018. DOI: 10.1007/s00493-016-3502-y.

[5] Ye, D., Yang, Y., Mandal, B. and Klein, D. J. Graph invertibility and median eigenvalues, *Linear Algebra and its Applications*, 513:304-323, 2017. DOI:10.1016/j.laa.2016.10.020.

