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On exponential stability for mixtures with non-constant coefficients

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Abstract. We consider the system modeling a mixture of three materials with frictional dissipation and we show the exponential stability of semigroup associated. We show that the corresponding semigroup is exponentially stable if and only if the imaginary axis is contained in the resolvent set of the infinitesimal generator. In particular this implies the lack of polynomial stability to the corresponding semigroup.

Keywords. Strong stability, Exponential stability, Semigroups C_0, Mixture of materials

1 Introduction

The theory of mixtures of solids has been widely investigated in the last decades, see for example [1] and [6]. Here we study the one dimensional model of a mixture of n interacting continuous with reference configuration over $[0, l]$. Let us denote by $u^1 := u^1(x_1, t),\ u^2 := u^2(x_2, t), u^3 := u^3(x_3, t)$ where $x_i \in [0, l]$. We assume that the particles under consideration occupy the same position at time $t = 0$, so that, $x = x_i$. This work is dedicated to characterize the exponential stability of the following mixture problem:

$$\begin{array}{rcl} \rho_1 u^1_{tt} & = & a_{11}u^1_{xx} + a_{12}u^2_{xx} + a_{13}u^3_{xx} - \xi_1(x)u^1_t - \xi_2(x)u^2_t + (\xi_1(x) + \xi_2(x))u^3_t, \\ \rho_2 u^2_{tt} & = & a_{12}u^1_{xx} + a_{22}u^2_{xx} + a_{23}u^3_{xx} - \xi_2(x)u^1_t - \xi_3(x)u^2_t + (\xi_2(x) + \xi_3(x))u^3_t, \\ \rho_3 u^3_{tt} & = & a_{13}u^1_{xx} + a_{23}u^2_{xx} + a_{33}u^3_{xx} + (\xi_1(x) + \xi_2(x))u^1_t + (\xi_2(x) + \xi_3(x))u^2_t \\ & & -(\xi_1(x) + 2\xi_2(x) + \xi_3(x))u^3_t, \end{array}$$

where $\rho_i > 0$, $\xi(\cdot) \in C^1(0, l)$, $\mathbf{A} = (a_{ij})$ and

$$\mathbf{D}(x) = \begin{bmatrix} \xi_1(x) & \xi_2(x) \\ \xi_2(x) & \xi_3(x) \end{bmatrix}$$

are positive definite (real) symmetric matrices. If denoted $\mathbf{C}$ as

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix},$$

we will have the following problem written in matrix form,

$$\mathbf{R}U_{tt} - \mathbf{A}U_{xx} + \mathbf{N}U + \mathbf{B}(x)U_t = 0, \tag{1}$$

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where $U = (u^1, u^2, u^3)^T$, $\mathbf{D}_0$ is a positive definite (real) symmetric matrix, $\mathbf{B} = \mathbf{C}^T\mathbf{D}(x)\mathbf{C}$ and $\mathbf{N} = \mathbf{C}^T\mathbf{D}_0\mathbf{C}$ are positive semidefinite (real) symmetric matrices. The initial conditions are given by

$$U(x, 0) = U_0(x), \quad U_t(x, 0) = U_1(x). \tag{2}$$

Finally, we consider Dirichlet boundary conditions

$$U(0, t) = U(l, t) = 0, \;, \quad t \in \mathbb{R}^+. \tag{3}$$

In [3] the authors showed that when $\mathbf{N} = 0$ and $\mathbf{B}$ is a constant matrix then the semigroup associated to (1)-(3) is exponentially stable if and only if

$$\dim \text{span}\{\mathbf{B}_{(j)}\mathbf{W}^m;\; m = 0, 1, \cdots n-1\} = n,$$

where $\mathbf{B}_{(j)}$ is a row vector of $\mathbf{B}$ and $\mathbf{W} = \mathbf{R}^{-1}\mathbf{A}$. Moreover they prove that the system in this case never is polynomially stable. In particular their result implies in the corresponding semigroup is exponential stable if and only if it is strongly stable (as in the finite dimensional case).

2 Semigroup formulation

Let us recall something results of linear algebra, whose proofs we refer to Bernstein [2].

Theorem 2.1. *If $0 \prec D \in \mathbb{R}^{r\times r}$ and $C \in \mathbb{R}^{r\times n}$, then*

- $C^T DC \succeq 0$.
- *Rank* $C^T DC =$ *Rank* C.

Definition. Let be $\mathbf{W} \in \mathbb{R}^{n\times n}$. The pair $(\mathbf{W}, \mathbf{C})$ is observable if

$$\dim \text{span}\left\{\mathbf{C}_j,\; \mathbf{C}_j\mathbf{W},\, \mathbf{C}_j\mathbf{W}^2,\; ...,\; \mathbf{C}_j\mathbf{W}^{n-1},\; j = 1, 2..., n\right\} = n. \tag{4}$$

The following proposition is known as the Hautus test for observability.

Theorem 2.2. *The pair $(\mathbf{W}, C)$ is observable if and only if*

$$rank \begin{bmatrix} \mathbf{W} - I\lambda \\ C \end{bmatrix} = n$$

for all λ eigenvalue of $\mathbf{W}$.

From now on we use the semigroup theory ($n = 3, r = 2$) to show the well posedness. To do that let us introduce the phase space $\mathcal{H}$

$$\mathcal{H} = [H_0^1(0, l)]^n \times [L^2(0, l)]^n,$$

that is a Hilbert space with the induzed norm

$$||(U, V)||^2_{\mathcal{H}} = \int_0^l U_x^*\mathbf{A}U_x dx + \int_0^l V^*\mathbf{R}V dx + \int_0^l U^*\mathbf{N}U dx$$

Let us introduce the operator $\mathcal{A}$ given by

$$\mathcal{A}\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} V \\ \mathbf{R}^{-1}\mathbf{A}U_{xx} - \mathbf{R}^{-1}\mathbf{N}U - \mathbf{R}^{-1}\mathbf{B}V \end{pmatrix} \tag{5}$$

with domain

$$D(\mathcal{A}) = [H_0^1(0,l) \cap H^2(0,l)]^n \times [H_0^1(0,l)]^n.$$

Under this conditions the initial-boundary value problem can be rewritten as

$$\frac{d}{dt}\mathbf{U} = \mathcal{A}\mathbf{U}, \quad \mathbf{U}(0) = \mathbf{U}_0$$

where $\mathbf{U}(t) = (U(t), V(t))^\top$ and $\mathbf{U}_0 = (U_0, U_1)^\top$.

Theorem 2.3. *The operator $\mathcal{A}$ is the infinitesimal generator of a contractions C_0-semigroup, we denote as $\mathcal{S}_\mathcal{A}(t) = e^{\mathcal{A}t}$.*

Proof. Note that $D(\mathcal{A})$ is dense in $\mathcal{H}$ and a dissipative operator $\mathcal{A}$, that is

$$Re\,(\mathcal{A}\mathbf{U}, \mathbf{U})_\mathcal{H} = -\int_0^l V^*\mathbf{B}V\,dx \leq 0. \tag{6}$$

Therefore we only need to show that $0 \in \rho(\mathcal{A})$ (see Liu and Zheng [5]). In fact, we prove that for any $\mathbf{F} = (F, G) \in \mathcal{H}$ there exists a unique $\mathbf{U} = (U, V)$ in $\mathcal{D}(\mathcal{A})$ such that $\mathcal{A}\mathbf{U} = \mathbf{F}$. In term of their components

$$V = F, \quad \mathbf{A}U_{xx} - \mathbf{N}U - \mathbf{B}(x)V = \mathbf{R}G \tag{7}$$

the above problem reduces to find $U \in [H^2 \cap H_0^1]^n$ such that

$$\mathbf{A}U_{xx} - \mathbf{N}U = \mathbf{B}(x)F + \mathbf{R}G.$$

But this problem is well posed and $\|\mathbf{U}\|_\mathcal{H} \leq C\|\mathbf{F}\|_\mathcal{H}$, so $0 \in \varrho(\mathcal{A})$.

□

Other important tool we use is the characterization of the exponential stability of a C_0 semigroup was obtained by Huang [4], and Pruss [7] independently. Here we use the version due to Pruss.

Theorem 2.4. *Let $\mathcal{S}_A(t)$ be a C_0-semigroup of contractions of linear operators on Hilbert space H with infinitesimal generator A. Then $\mathcal{S}_A(t)$ is exponentially stable if and only if*

$$i\mathbb{R} \subset \rho(A) \tag{8}$$

and

$$\limsup_{|\lambda|\to+\infty} ||(i\lambda I - A)^{-1}||_{\mathcal{L}(H)} < \infty \tag{9}$$

where $\mathcal{L}(H)$ denotes the space of continuous linear functions in H.

Note that $\mathcal{A}^{-1}$ defined in (5) is compact so $D(\mathcal{A})$ is compactly embedded into $\mathcal{H}$. Thus we conclude that the spectrum of the operator $\mathcal{A}$ consists entirely of isolated eigenvalues.

The resolvent equation can be written as

$$i\lambda\mathbf{U} - \mathcal{A}\mathbf{U} = \mathbf{F}. \tag{10}$$

where $\mathbf{U} = (U, V) \in D(\mathcal{A})$, $\mathbf{F} = (F, G) \in \mathcal{H}$ and $\lambda \in \mathbb{R}$. Taking the inner product in $\mathcal{H}$ and considering the real part we get

$$\int_0^l V^*\mathbf{B}V\,dx = \text{Re }(\mathbf{U}, \mathbf{F})_\mathcal{H}. \tag{11}$$

3 On the stability of the system

In this section we assumed that $\mathbf{D} \succeq \mathbf{D}_0 \succ 0$. Note that Rank$\mathbf{B}(x)$ is constant (Theorem 2.1). We first see that in terms of the components the resolvent equation (10) can be written as

$$i\lambda U = V + F \tag{12}$$

$$i\lambda V = \underbrace{\mathbf{R}^{-1}\mathbf{A}}_{=\mathbf{W}} U_{xx} - \mathbf{R}^{-1}\underbrace{\mathbf{C}^T\mathbf{D}_0\mathbf{C}}_{=\mathbf{N}} U - \mathbf{R}^{-1}\underbrace{\mathbf{C}^T\mathbf{D}(x)\mathbf{C}}_{=\mathbf{B}(x)} V + G. \tag{13}$$

The next Lemma will play an important role in the sequel.

Lemma 3.1. *The operator $\mathcal{A}$ satisfies the condition (8) if and only if (4) holds.*

Proof. Let us suppose that (8) is false, then there exists $0 \neq \mathbf{U} \in D(\mathcal{A})$ satisfying (10) with $\mathbf{F} = 0$. Using (11) we get

$$\int_0^l V^*\mathbf{B}V\,dx = 0 \quad \Longrightarrow \quad V^*(x)\mathbf{B}(x)V(x) = 0\ , \quad \text{almost always in } \ (0,l). \tag{14}$$

Note that in this case we have that $U = (u^1, u^2 \cdots u^n)^T$ and u^i must be of the form

$$u^i = \alpha^i_\nu \sin\left(\frac{\nu\pi}{l}x\right), \nu \in \mathbb{N}.$$

Because this functions verifies the boundary conditions (3). Note that the above functions u^i are the eigenvalues of the problem

$$\begin{cases} \quad -w_{xx} \ = \ \mu w \\ w(0) \ = \ w(l) \ = \ 0, \quad \mu = \left(\frac{\nu\pi}{l}\right)^2, \end{cases}$$

and moreover this is a basis of the $L^2(0,l)$, moreover it is a basis of $\{f \in H^1(0,l),\ f(0) = f(l) = 0\}$. This means that $U = Y_\nu \sin\left(\frac{\nu\pi}{l}x\right)$ where $Y_\nu = (\alpha^1_\nu, \cdots \alpha^n_\nu)^T$ and $V = i\lambda Y \sin(\frac{n\pi x}{l})$. Then

$$\lambda^2 \sin^2(\frac{n\pi x}{l})[\mathbf{C}Y]^*\mathbf{D}(x)[\mathbf{C}Y] = 0.$$

Since $\mathbf{D} \succeq \mathbf{D}_0$ we have

$$\lambda^2 \sin^2(\frac{n\pi x}{l})[\mathbf{C}Y]^*\mathbf{D}_0[\mathbf{C}Y] = 0.$$

Since $\mathbf{D}_0$ is a positive definite we have $\mathbf{C}Y = 0$. Then

$$\mathbf{C}U = 0. \tag{15}$$

Now, (12)-(13) is equivalent to

$$-\lambda^2 U = \mathbf{W}U_{xx}, \quad \mathbf{C}U = 0. \tag{16}$$

So, we have

$$\mathbf{CW}U_{xx} = 0 \quad \Rightarrow \quad \mathbf{CW}U = 0.$$

multiplying by $\mathbf{CW}$ the first equation in (16) we get $\mathbf{CW}^2U = 0$. Using induction we get that $\mathbf{CW}^mU = 0$ for all m. If (4) holds then $U = 0$ which is a contradiction, therefore (4) is not true. To prove the other implication, note that, if (4) is false then there exists $Y \neq 0$ such that $Y \in \mathbb{R}^n \backslash \{0\}$ is such that

$$\mathbf{C}Y = 0, \quad (\mathbf{W} - \tau I)\, Y = 0; \quad \tau > 0 \tag{17}$$

see theorem 2.2. Then the functions

$$\mathbf{U}_\nu = \Big(\sin(\frac{\nu\pi}{l}x)Y \ , \ i\lambda_\nu \sin(\frac{\nu\pi}{l}x)Y\Big) \in D(\mathcal{A}), \ \nu \in \mathbb{N}, \tag{18}$$

are the eigenvectors of $\mathcal{A}$ with $i\lambda_\nu = i\frac{\nu\pi}{l}\sqrt{\tau}$ the corresponding imaginary eigenvalues, for $\nu \in \mathbb{N}$. Therefore $i\mathbb{R} \cap \sigma(\mathcal{A}) \neq \emptyset$.

□

Note that (11) implies

$$\int_0^l |\mathbf{C}V|^2 \lambda_1(x) dx \leq C||\mathbf{U}||_{\mathcal{H}} ||\mathbf{F}||_{\mathcal{H}} \tag{19}$$

where $\lambda_1(x)$ is the first eigenvalue of $\mathbf{D}(x)$.

Lemma 3.2. *The operator $\mathcal{A}$ satisfies the condition (9) if (4) is holds.*

Proof. Suppose that (4) is not true. There then exists a sequence ω_ν with $\omega_\nu \to \infty$ and a sequence vectors functions $\mathbf{U}_\nu = (U_\nu, V_\nu) \in D(\mathcal{A})$ with unit norm in $\mathcal{H}$ such that as $\nu \to \infty$,

$$(i\omega_\nu I - \mathcal{A})\, \mathbf{U}_\nu \longrightarrow 0, \quad \text{in } \mathcal{H},$$

This is

$$F_\nu = i\omega_\nu U_\nu - V_\nu \to 0 \text{ in } \left[H_0^1(0,l)\right]^n \tag{20}$$

$$\mathbf{R}G_\nu = i\omega_\nu \mathbf{R}V_\nu - \mathbf{A}U_{\nu,xx} + \mathbf{N}U_\nu + \mathbf{B}V_\nu \to 0 \text{ in } \left[L^2(0,l)\right]^n. \tag{21}$$

Taking the inner product of $(i\omega_\nu I - \mathcal{A})\, \mathbf{U}_\nu$ by $\mathbf{U}_\nu$ in $\mathcal{H}$ and using the estimate (20) yields that

$$\int_0^l |\mathbf{D}^{1/2}\mathbf{C}V_\nu|^2 \ dx = \int_0^l V_\nu^* \mathbf{B} V_\nu \ dx \ \to \ 0 \tag{22}$$

then

$$\int_0^l \lambda_1(x)|\mathbf{C}V_\nu|^2 \ dx \ \leq \ \int_0^l [\mathbf{C}V_\nu]^* \mathbf{D}(x)[\mathbf{C}V_\nu] \ dx \ \ \to \ 0. \tag{23}$$

On other hand, we can easily deduce from (21) that

$$-\omega_\nu^2 \mathbf{R}U_\nu - \mathbf{A}U_{\nu,xx} \ = \ i\omega_\nu \mathbf{R}F_\nu + \mathbf{R}G_\nu - \mathbf{N}U_\nu - \mathbf{B}V_\nu. \tag{24}$$

step 1 Multiplying equation (24) by $\mathbf{C}\mathbf{W}^{-1}\mathbf{R}^{-1}$ we get

$$-\omega_\nu^2 \mathbf{C}\mathbf{W}^{-1}U_\nu - \mathbf{C}U_{\nu,xx} \ = \ i\omega_\nu \mathbf{C}\mathbf{W}^{-1}F_\nu + \mathbf{C}\mathbf{W}^{-1}G_\nu - \mathbf{C}\mathbf{A}^{-1}\mathbf{N}U_\nu - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}V_\nu.$$

and multiplying the above equation by $\lambda_1(x)(\mathbf{C}U_\nu)^*$ we obtain

$$-\omega_\nu \int_0^l \lambda_1(\omega_\nu \mathbf{C}U_\nu)^* \mathbf{C}\mathbf{W}^{-1}U_\nu \ dx + \int_0^l (\lambda_1 \mathbf{C}U_{\nu,x})^* \mathbf{C}U_{\nu,x} \ dx \ = \tag{25}$$

$$i\omega_\nu \int_0^l \lambda_1(\mathbf{C}U_\nu)^* \mathbf{C}\mathbf{W}^{-1}F_\nu \ dx + \int_0^l \lambda_1(\mathbf{C}U_\nu)^* \mathbf{C}\mathbf{W}^{-1}G_\nu \ dx - \int_0^l (\lambda_1' \mathbf{C}U_\nu)^* \mathbf{C}U_{\nu,x} \ dx$$

$$-\int_0^l \lambda_1(\mathbf{C}U_\nu)^*\mathbf{C}\mathbf{A}^{-1}\mathbf{N}U_\nu\ dx - \int_0^l \lambda_1(\mathbf{C}U_\nu)^*\mathbf{C}\mathbf{A}^{-1}\mathbf{B}V_\nu\ dx.$$

Since $U_{\nu,x}$ and $\omega_\nu U_\nu$ are uniformly bounded in $[L^2(0,l)]^n$, the terms on the right hand side of (25) converge to zero. From (20) we deduce that the first term the left hand side of (25) converge to zero. Then (25) and (23)implies

$$\int_0^l |\mathbf{C}V_\nu|^2\lambda_1(x) + |\mathbf{C}U_{\nu,x}|^2\lambda_1(x)\ dx \longrightarrow 0\ ,\ \text{as}\ \nu \longrightarrow +\infty. \tag{26}$$

Step 2 Using a procedure indutive, multiplying equation (24) by $\lambda_1(x)\mathbf{C}\mathbf{W}^{m-1}\mathbf{R}^{-1}$ we get

$$\int_0^l |\mathbf{C}\mathbf{W}^m V|^2\lambda_1(x)\ +\ |\mathbf{C}\mathbf{W}^m U_x|^2\lambda_1(x)\ dx \longrightarrow 0\ ,\ \text{as}\ \nu \longrightarrow +\infty. \tag{27}$$

for $m = 0, ..., n-1$. Using the hypotheses we get

$$\int_0^l |V_\nu|^2\lambda_1(x)\ +\ |U_{\nu,x}|^2\lambda_1(x)\ dx \longrightarrow 0\ ,\ \text{as}\ \nu \longrightarrow +\infty.$$

Since $\mathbf{A}, \mathbf{R} \succ 0$ we obtain

$$\int_0^l V_\nu^*\mathbf{R}V_\nu\lambda_1(x)\ +\ U_{\nu,x}^*\mathbf{A}U_{\nu,x}\lambda_1(x)\ dx \longrightarrow 0\ ,\ \text{as}\ \nu \longrightarrow +\infty. \tag{28}$$

Step 3 Let $q(x)$ be a real function C^1 wich will be chosen later. Let $\mathbf{Q} \succ 0$ a real symmetric matrix. Note that for $W \in H^1(0,l)$, we have

$$2\ \mathrm{Re}\int_0^l q(x)W^*\mathbf{Q}W_x\ dx\ =\ q(l)W^*(l)\mathbf{Q}W(l)\ -\ q(0)W^*(0)\mathbf{Q}W(0) - \int_0^l q'(x)W^*\mathbf{Q}W\ dx. \tag{29}$$

Taking the inner product of (24) with $q(x)U_{\nu,x}^*$, integrating by parts , we obtain

$$-\int_0^l q(x)[\omega_\nu^2 U_{\nu,x}^*\mathbf{R}U_\nu + U_{\nu,x}^*\mathbf{A}U_{\nu,xx}]\ dx\ =\ \int_0^l [i\omega_\nu U_\nu]^*[\mathbf{R}F_\nu q(x)]_x\ dx$$

$$+\int_0^l q(x)U_{\nu,x}^*\mathbf{R}G_\nu\ dx - \int_0^l q(x)U_{\nu,x}^*\mathbf{N}U_\nu\ dx - \int_0^l q(x)U_{\nu,x}^*\mathbf{B}V_\nu\ dx,$$

and using (29), we obtain that

$$\int_0^l q'(x)[\omega_\nu^2 U_\nu^*\mathbf{R}U_\nu + U_{\nu,x}^*\mathbf{A}U_{\nu,x}]\ dx - q(x)U_{\nu,x}^*\mathbf{A}U_{\nu,x}|_0^l\ = \tag{30}$$

$$2\mathrm{Re}\ \int_0^l [i\omega_\nu U_\nu]^*[\mathbf{R}F_\nu q(x)]_x\ dx +\ 2\mathrm{Re}\ \int_0^l q(x)U_{\nu,x}^*\mathbf{R}G_\nu\ dx$$
$$-\ 2\mathrm{Re}\ \int_0^l q(x)U_{\nu,x}^*\mathbf{N}U_\nu\ dx - 2\mathrm{Re}\ \int_0^l q(x)U_{\nu,x}^*\mathbf{B}V_\nu\ dx.$$

Since $U_{\nu,x}$ and $\omega_\nu U_\nu$ are uniformly bounded in $[L^2(0,l)]^n$, the terms on the right hand side of (30) converge to zero.

Taking $q(x) = x$ we deduce from (30) and $||(U_\nu, V_\nu)||^2_{\mathcal{H}} = 1$ that

$$U^*_{\nu,x}(l)\mathbf{A}U_{\nu,x}(l) \longrightarrow \frac{1}{l} \ ; \quad \text{as} \quad \nu \longrightarrow \infty. \tag{31}$$

Taking $q(x) = l - x$ we deduce from (30) and $||(U_\nu, V_\nu)||^2_{\mathcal{H}} = 1$ that

$$U^*_{\nu,x}(0)\mathbf{A}U_{\nu,x}(0) \longrightarrow \frac{1}{l} \ ; \quad \text{as} \quad \nu \longrightarrow \infty. \tag{32}$$

We now take $q(x) = \displaystyle\int_0^x \lambda_1(s)\ ds$ in (30) to obtain that

$$\int_0^l [V^*_\nu \mathbf{R} V_\nu + U^*_{\nu,x}\mathbf{A}U_{\nu,x}]\lambda_1(x)\ dx \longrightarrow \ \overline{\lambda}_1 = \frac{1}{l}\int_0^l \lambda_1(x)\ dx > 0\ . \tag{33}$$

this is contradiction with (28).

□

4 Conclusions

As a consequence of the above result we have that the following statements are equivalents:

- $\mathcal{S}_{\mathcal{A}}(t)$ is strongly stable
- dim span $\left\{\mathbf{C}_j,\ \mathbf{C}_j\mathbf{W},\ \mathbf{C}_j\mathbf{W}^2,\ ...,\ \mathbf{C}_j\mathbf{W}^{n-1},\ j = 1, 2..., n\right\} = n.$
- $\mathcal{S}_{\mathcal{A}}(t)$ is exponentially stable

In particular our result implies that the corresponding semigroup is exponential stable if and only if it is strongly stable (as in the finite dimensional case).

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