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On new results about partitions into parts congruent to $\pm 1 \pmod 5$

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Abstract. In this work we talk about some patterns on partitions considered by the 1^{st} Rogers-Ramanujan Identity. Looking for a new bijective proof for it, we have studied partitions into parts congruent to $\pm 1 \pmod 5$ and have created a two-line matrix representation for them. By adding up their second line elements, we have obtained the number of parts of the related partitions. We classify the partitions according to the sum on the second row of the matrix associated to it and organize the data on a table, obtaining some partition identitities.

Keywords. Partitions, Rogers-Ramanujan's Identities, Congruences.

1 Introduction

One of the most known identities in Partition Theory, due to Srinivasa Ramanujan (at the same time and independently, Leonard Rogers), called the 1^{st} Rogers-Ramanujan Identity, says that the number of partitions of a given n into parts congruent to ± 1 (mod 5) is equal to the number of partitions of n into 2-distinct parts. An analitic proof is presented in [1] and in [2], there is a bijective proof for this identity, although it is not simple. So, as we found in [3] a two-line matrix representation for partitions into 2-distinct parts, if we could get a matrix representation for the other set of partitions, maybe we could get a bijective proof for the 1^{st} Rogers-Ramanujan Identity. Even though we are not able to get this proof yet, we got other results provided by this matrix representation. Some of them are presented in this work.

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2 Some Results

We start by showing a new two-line matrix representation for partitions whose parts are congruent to $\pm 1 \pmod 5$.

Theorem 2.1. *The number of partitions of n into parts congruent to $\pm 1 \pmod 5$ is equal to the number of two-line matrices*

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix}, \tag{1}$$

whose entries satisfy the following relations:

$$\begin{aligned} c_s &= c_{s-1} = 0, && (2)\\ d_t &\geq 0, && (3)\\ c_{2i-1} &= 5d_{2i+1} + 5d_{2i+3} + \ldots, && (4)\\ c_{2i} &= 5\frac{d_{2i+2}}{4} + 5\frac{d_{2i+4}}{4} + \ldots, && (5)\\ n &= \sum c_t + \sum d_t. && (6) \end{aligned}$$

Proof. Let R_k be the k-th natural number congruent to $\pm 1 \pmod 5$, i.e.,

$$\begin{aligned} R_{2i-1} &= 5 \cdot (i-1) + 1 \\ R_{2i} &= 5 \cdot (i-1) + 4 \end{aligned}$$

Given a partition of n into parts congruent to $\pm 1 \pmod 5$, let us suppose R_s is the largest part of the partition, with s an even number, saying,

$$\begin{aligned} n &= j_1 \cdot R_1 + j_2 \cdot R_2 + \cdots + j_s \cdot R_s, \qquad \text{where } j_k = \begin{cases} d_k & \text{for odd } k \\ \frac{d_k}{4} & \text{for even } k \end{cases} \\ &= j_1(5 \cdot 0 + 1) + j_2(5 \cdot 0 + 4) + j_3(5 \cdot 1 + 1) + j_4(5 \cdot 1 + 4) + \cdots + j_s(5 \cdot \left(\frac{s-2}{2}\right) + 4) \\ &= 5(0 \cdot j_1 + 1 \cdot j_3 + 2 \cdot j_5 + \cdots + \left(\frac{s-2}{2}\right) \cdot j_{s-1}) + (j_1 + j_3 + j_5 + \cdots + j_{s-1}) \\ &\quad + 5(0 \cdot j_2 + 1 \cdot j_4 + 2 \cdot j_6 + \cdots + \left(\frac{s-2}{2}\right) \cdot j_s) + 4(j_2 + j_4 + j_6 + \cdots + j_s) \end{aligned}$$

It is easy to see that we can associate the partition above to an unique matrix of type (1), satisfying conditions (2) to (6).

Conversely, by summing the entries of any matrix of type (1) satisfying the same conditions we get an unique partition of n into parts congruent to $\pm 1 \pmod 5$.

The case with s an odd number is analogous.

□

The second row of those matrices above describes the number of parts of the partition associated to it. For a fixed n, we classify its partitions into parts congruent to $\pm 1 \pmod 5$ according to the sum of entries d_i, for odd i, and $\frac{d_i}{4}$, for even i, on the second row of the matrix associated to each one of them. By counting the appearance of each number in these sums, we can organize the data on a table, which is presented below. The entry in line n and column $n-j$ is the number of times j appears as sum of the entries of the second row in type (1) matrices.

Table 1: Table from the characterization given by Theorem (2.1)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
1	1																													
2	1	0																												
3	1	0	0																											
4	1	0	0	1																										
5	1	0	0	1	0																									
6	1	0	0	1	0	1																								
7	1	0	0	1	0	1	0																							
8	1	0	0	1	0	1	1	0																						
9	1	0	0	1	0	1	1	0	1																					
10	1	0	0	1	0	1	1	0	2	0																				
11	1	0	0	1	0	1	1	0	2	0	1																			
12	1	0	0	1	0	1	1	0	2	1	2	0																		
13	1	0	0	1	0	1	1	0	2	1	2	1	0																	
14	1	0	0	1	0	1	1	0	2	1	2	2	0	1																
15	1	0	0	1	0	1	1	0	2	1	2	2	0	3	0															
16	1	0	0	1	0	1	1	0	2	1	2	2	1	4	0	1														
17	1	0	0	1	0	1	1	0	2	1	2	2	1	4	1	2	0													
18	1	0	0	1	0	1	1	0	2	1	2	2	1	4	2	3	2	0												
19	1	0	0	1	0	1	1	0	2	1	2	2	1	4	2	3	4	0	1											
20	1	0	0	1	0	1	1	0	2	1	2	2	1	4	2	4	5	0	4	0										
21	1	0	0	1	0	1	1	0	2	1	2	2	1	4	2	4	5	1	6	0	1									
22	1	0	0	1	0	1	1	0	2	1	2	2	1	4	2	4	5	2	7	2	3	0								
23	1	0	0	1	0	1	1	0	2	1	2	2	1	4	2	4	5	2	7	4	4	2	0							
24	1	0	0	1	0	1	1	0	2	1	2	2	1	4	2	4	5	2	8	5	5	6	0	1						
25	1	0	0	1	0	1	1	0	2	1	2	2	1	4	2	4	5	2	8	5	6	8	0	5	0					
26	1	0	0	1	0	1	1	0	2	1	2	2	1	4	2	4	5	2	8	5	7	9	2	9	0	1				
27	1	0	0	1	0	1	1	0	2	1	2	2	1	4	2	4	5	2	8	5	7	9	4	11	3	3	0			
28	1	0	0	1	0	1	1	0	2	1	2	2	1	4	2	4	5	2	8	5	7	10	5	12	7	5	3	0		
29	1	0	0	1	0	1	1	0	2	1	2	2	1	4	2	4	5	2	8	5	7	10	5	13	9	6	9	0	1	
30	1	0	0	1	0	1	1	0	2	1	2	2	1	4	2	4	5	2	8	5	7	10	5	14	10	9	14	0	6	0

Definition 2.1. *Let $p_{\pm 1(5)}(n,k)$ be the number of partitions of n into k parts congruent to $\pm 1 \pmod 5$ and $P_{\pm 1(5)}(n,k)$ the set of all the partitions counted by $p_{\pm 1(5)}(n,k)$. So, $|P_{\pm 1(5)}(n,k)| = p_{\pm 1(5)}(n,k)$.*

By observing the table above we can see that the columns become constant from certain values of n on. This is summarized next.

Proposition 2.1. *For all $n \geq 2$ and $i \geq 0$ we have*

(i) $p_{\pm 1(5)}(4n-2, n-1) = p_{\pm 1(5)}(4n-2+i, n-1+i)$

(ii) $p_{\pm 1(5)}(4n, n) = p_{\pm 1(5)}(4n+i, n+i)$

(iii) $p_{\pm 1(5)}(4n, n-1) = p_{\pm 1(5)}(4n+i, n-1+i)$

The following result was also suggested by the table.

Proposition 2.2. *For all $n \geq 1$,*

$$p_{\pm 1(5)}(5n-3,2) = p_{\pm 1(5)}(5n+3,2) = \left\lfloor \frac{n+1}{2} \right\rfloor.$$

Proof. Let (λ_1, λ_2) be a partition of $5n-3$ into two parts congruent to $\pm 1 \pmod 5$. Note that both parts need to be congruent to $1 \pmod 5$. On the other hand, a partition (μ_1, μ_2) of $5n+3$ into two parts congruent to $\pm 1 \pmod 5$ has to have both of them congruent to $-1 \pmod 5$. Clearly, $(\lambda_1 + 3, \lambda_2 + 3)$ leads us to a partition of $5n+3$ and $(\mu_1 - 3, \mu_2 - 3)$ to a partition of $5n-3$.

To prove the second equality, let us consider a partition (μ_1, μ_2) of $5n+3$, that is, $(5r-1) + (5s-1) = 5n+3, \quad \text{with } r, s \geq 1$, which is the same as finding the number of solutions of the previous equation, without counting the order. Or, $r + s = n + 1$.

Then we have $p_{\pm 1(5)}(5n+3,2) = p(n+1,2) = \left\lfloor \frac{n+1}{2} \right\rfloor$. □

3 Main Theorem

Frequently, in our work, we note the appearance of Triangular Numbers. By observing the fourth diagonal, they also show up in some partitions into four parts. For those ones, we got a closed formula to count them.

Theorem 3.1. *Being T_n the n-th triangular number, for all $n \geq 0$,*

(i) $p_{\pm 1(5)}(10n,4) = 2 \cdot \sum_{i=1}^{n-1} T_i + T_n = \frac{(n+1)n(2n+1)}{6}$;

(ii) $p_{\pm 1(5)}(10n+5,4) = 2 \cdot \sum_{i=1}^{n} T_i = \binom{n+2}{3}$.

In order to prove the Main Theorem, we need some other statements, as follows. Some proofs we omit.

Lemma 3.1. *For all $n \geq 0$,*

(i) The number of partitions of $10n+1$ into three parts congruent do $\pm 1 \pmod 5$ whose difference between the two largest parts is less than 5, is equal to n;

(ii) The number of partitions of $10n+6$ into three parts congruent do $\pm 1 \pmod 5$ whose difference between the two largest parts is less than 5, is equal to $n+1$.

Sketch of the proof: (i) As we always have two parts congruent to $1 \pmod 5$ and one to $4 \pmod 5$, we can classify the partitions according to the position occupied by its part congruent to $4 \pmod 5$. Then we have three possible values for differences between the two largest parts: 0, 2 and 3. In each case, we study the number of solutions of a equation into 2 variables and it gives us the desired result. Item (ii) has an analogous proof.

Proposition 3.1. *For all $n \geq 0$,*

(i) $p_{\pm 1(5)}(10n+1,3) - p_{\pm 1(5)}(10n-4,3) = n$;

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(ii) $p_{\pm 1(5)}(10n+6,3) - p_{\pm 1(5)}(10n+1,3) = n+1$.

Proof. We prove item (i), the other one being similar.

Let us write $P_{\pm 1(5)}(n,k) = P_{\pm 1(5)}^{\lambda_1-\lambda_2\geq 5}(n,k) \cup P_{\pm 1(5)}^{\lambda_1-\lambda_2<5}(n,k)$, where $P_{\pm 1(5)}^{\lambda_1-\lambda_2\geq 5}(n,k)$ is the subset of partitions whose difference between the two largest part is equal to or greater than 5 and $P_{\pm 1(5)}^{\lambda_1-\lambda_2<5}(n,k)$, less than 5. It's easy to see that there's a bijection between $P_{\pm 1(5)}^{\lambda_1-\lambda_2\geq 5}(10n+1,3)$ and $P_{\pm 1(5)}(10n-4,3)$. By Lemma (3.1), we have $|P_{\pm 1(5)}^{\lambda_1-\lambda_2<5}(10n+1,3)| = n$, and the theorem is proved. □

From Proposition (3.1), we get the following corollary.

Corollary 3.1. *For all $n \geq 0$,*

(i) $p_{\pm 1(5)}(10n+1,3) = 2T_n$*;*

(ii) $p_{\pm 1(5)}(10n+6,3) = T_n + T_{n+1}$.

Proof.

(i) Combining both items of Proposition 3.1 we have

$$p_{\pm 1(5)}(10n+1,3) \quad - \quad p_{\pm 1(5)}(10n-4,3) = n \tag{7}$$
$$p_{\pm 1(5)}(10(n-1)+6,3) \quad - \quad p_{\pm 1(5)}(10(n-1)+1,3) = n. \tag{8}$$

On equations (7) and (8), if we change n by $n-1$, $n-2$, …, 2 and 1 and add all the equations up, we get

$$p_{\pm 1(5)}(10n+1,3) = 2n + 2(n-1) + 2(n-3) + \ldots + 2 = 2T_n.$$

(ii) Combining item (i) of this corollary and Proposition 3.1, we get

$$p_{\pm 1(5)}(10n+6,3) = 2T_n + (n+1) = T_n + T_{n+1}$$

□

Proposition 3.2. *For all $n \geq 0$,*

(i) $p_{\pm 1(5)}(10n+1,3) = p_{\pm 1(5)}(10n+4,3)$*;*

(ii) $p_{\pm 1(5)}(10n+6,3) = p_{\pm 1(5)}(10n+9,3)$.

Proof. The same bijection holds for both items. As an example, we take the second one. Note that any partition of $10n+6$ into 3 parts congruent to $\pm 1 \pmod 5$ has to have two parts congruent to $1 \pmod 5$ and one part congruent to $4 \pmod 5$. And any partition of $10n+9$ into 3 parts congruent to $\pm 1 \pmod 5$ has to have two parts congruent to $4 \pmod 5$ and one part congruent to $1 \pmod 5$.

So, a bijection between the two sets of partitions adds 3 to the smallest part congruent to $1 \pmod 5$ of any partition of $10n+6$ and subtracts 3 of the smallest part congruent to $4 \pmod 5$ of any partition of $10n+9$. Note that there may not always be the smallest part. However, this does not affect the bijection.

□

Finally, before being able to prove our Main Theorem, we need one last result.

Lemma 3.2. *For all $n \geq 0$ and $i \leq n$,*

(i) $p_{\pm 1(5)}(10n, 4,$ *smallest part* $R_{2i-1}) = p_{\pm 1(5)}(10(n-(2i-1))+9, 3)$;

(ii) $p_{\pm 1(5)}(10n+5, 4,$ *smallest part* $R_{2i-1}) = p_{\pm 1(5)}(10(n-2(i-1))+1, 3)$;

(iii) $p_{\pm 1(5)}(10n, 4,$ *smallest part* $R_{2i}) = p_{\pm 1(5)}(10(n-2i)+6, 3)$;

(iv) $p_{\pm 1(5)}(10n+5, 4,$ *smallest part* $R_{2i}) = p_{\pm 1(5)}(10(n-(2i-1))+1, 3)$.

Proof. For our purpose, the proofs we show are from the first and third items, being the second and fourth similar.

(i) Taking a partition of $10n$ into 4 parts whose smallest part is R_{2i-1}, we can write $10n$ as

$$10n = \lambda_1 + \lambda_2 + \lambda_3 + 5(i-1) + 1,$$

which is the same as

$$10n - 5(i-1) - 1 = \lambda_1 + \lambda_2 + \lambda_3.$$

As all parts are greater than or equal to $R_{2i-1} = 5(i-1)+1$, we may decrease each λ_j by $5(i-1)$, then counting partitions of $10(n-(2i-1))+9$ into 3 parts congruent to $\pm 1 \pmod 5$, so

$$p_{\pm 1(5)}(10(n-(2i-1))+9, 3).$$

(iii) Let $(\lambda_1, \lambda_2, \lambda_3, R_{2i})$ be a partition lying on $P_{\pm 1(5)}(10n, 4,$ smallest part $R_{2i})$. In the same way we did before, we remove the part R_{2i}, decrease the remaining congruent to 4 (mod 5) by $5(i-1)$ and the congruent to 1 (mod 5) by $5i$. Note that it is possible since these last parts are at least 6.

The next table illustrates the case where $n = 5$ and $i = 2$.

Table 2: Example for Lemma 3.2.(*iii*) with $n = 5$ and $i = 2$.

$P_{\pm 1(5)}(50, 4,$ smallest part $R_4)$			$P_{\pm 1(5)}(16, 3)$
$(21, 11, 9, 9)$	$(21, 11, 9)$	$(11, 1, 4)$	$(11, 4, 1)$
$(19, 11, 11, 9)$	$(19, 11, 11)$	$(14, 1, 1)$	$(14, 1, 1)$
$(16, 16, 9, 9)$	$(16, 16, 9)$	$(6, 6, 4)$	$(6, 6, 4)$
$(16, 14, 11, 9)$	$(16, 14, 11)$	$(6, 9, 1)$	$(9, 6, 1)$

□

After all the results we have presented, Theorem 3.1 becomes easy to be proved and it goes as follows.

Proof of Theorem 3.1: Remembering that R_i is the $i-th$ number congruent to $\pm 1 \pmod 5$. We are going to classify all partitions counted by $p_{\pm 1(5)}(10n, 4)$ according to its smallest part. Note that the maximum value for this part is R_n, which could be

$$R_{2i} = 5(i-1)+4, \text{ for } i \leq \left\lfloor \frac{n}{2} \right\rfloor \quad \text{or} \quad R_{2i-1} = 5(i-1)+1, \text{ for } i \leq \left\lfloor \frac{n+1}{2} \right\rfloor.$$

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(i) After this classification, we have

$$\begin{aligned} p_{\pm 1(5)}(10n,4) &= \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} p_{\pm 1(5)}(10n,4,\ \text{smallest part } R_{2i-1}) \\ &+ \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} p_{\pm 1(5)}(10n,4,\ \text{smallest part } R_{2i}). \end{aligned}$$

By items (i) and (iii) of Lemma 3.2, this equality turns to

$$\begin{aligned} p_{\pm 1(5)}(10n,4) &= \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} p_{\pm 1(5)}(10(n-(2i-1))+9,3) \\ &+ \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} p_{\pm 1(5)}(10(n-2i)+6,3). \end{aligned}$$

By combining item (ii) of Proposition 3.2 and item (ii) of Corollary 3.1, we are allowed to write

$$\begin{aligned} p_{\pm 1(5)}(10n,4) &= \sum_{i=1}^{n} p_{\pm 1(5)}(10(n-i)+6,3) \\ &= \sum_{i=1}^{n} (T_{n-i}+T_{n-i+1}) \quad = \quad 2\sum_{i=1}^{n-1} T_i + T_n. \end{aligned}$$

The second equality follows by Induction.

(ii) The proof is analogous to the previous one, just using items (ii) and (iv) of Lemma 3.2 and item (i) of Corollary 3.1 instead of those used before.

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