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Advances in a Hypergraph Coloring Conjecture

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Abstract. We consider a conjecture introduced by Hoppen, Kohayakawa and Lefmann. For fixed positive integers k, q and t with $1 \leq t < k$ and a k-uniform hypergraph H, let $\kappa(H, q, t)$ denote the number of q-colorings of the set of hyperedges of H for which any two hyperedges in the same color class intersect in at least t elements. Consider the function $\mathrm{KC}(n, k, q, t) = \max_{H \in \mathcal{H}_n} \kappa(H, q, t)$, where the maximum runs over the family $\mathcal{H}_{n,k}$ of all k-uniform hypergraphs on n vertices. Hoppen, Kohayakawa and Lefmann found the hypergraph H which satisfies $\kappa(H, q, t) = \mathrm{KC}(n, k, q, t)$ when $q \leq 4$ or $k \geq 2t - 1$. They proposed a conjecture when $q \geq 5$ and $k < 2t - 1$. In this work we proved this conjecture for $q \leq 9$.

Key-words. Extremal Set Theory, Hypergraphs, Colorings

1 Introdução

In Extremal Combinatorics, we are typically concerned with the largest (or smallest) structure satisfying some property P. For example, a classical problem for set systems is finding the largest family of subsets of $[n] = \{1, \ldots, n\}$ such that no set is contained in another (Sperner [8]). Another classical example, for graphs, is finding the graph with n vertices and with the largest number of edges not containing a fixed graph F as a subgraph. This problem is called the Turán Problem. Turán [9] found that, for $n \geq k$ and $F = K_k$, the graph with the largest number of edges is the balanced complete $(k-1)$-partite graph $T_{k-1}(n)$ with n vertices, also called Turán graph. In general, we say that a structure is optimum with respect to P if it is largest (or smallest) satisfying P. In this paper, we follow the notation of [3].

Erdős and Rothschild [4] were interested in a colored version of this problem. A q-coloring of a graph is a function associating every edge of the graph with a color in $[q]$, and it is said to be F-free if there is no monochromatic copy of F as a subgraph of G. When F is a path P_3 on three vertices, a P_3-free coloring is just a proper edge-coloring, in which distinct edges that share a vertex are assigned distinct colors. Erdős and Rothschild asked, for fixed q and k and a large enough n, for the graph n-vertex G with the maximum number of F-free q-colorings when $F = K_k$. In the context of coloring problems, a structure is

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optimum when it admits the largest number of colorings with the required property. A natural candidate to be optimum is $T_{k-1}(n)$, because we can color its edges freely, leading to $q^{|E(T_{k-1}(n))|}$ distinct colorings. For $q \in \{2,3\}$, Alon, Balogh, Keevash and Sudakov [2] (see also Yuster [10]) found that in fact $T_{k-1}(n)$ is the optimum graph, but for $q \geq 4$ this is not true. Recently, Pikhurko and Yilma [7] solved this problem for $q = 4$ and $F = K_3$ or $q = 4$ and $F = K_4$ when n is sufficiently large. However, for $q \geq 5$, or $q = 4$ with $F = K_k$ for $k \geq 5$, the optimum graphs are not known.

Hoppen, Kohayakawa and Lefmann [6] considered a colored version of the Erdős-Ko-Rado Theorem [5]. As usual, a hypergraph $H = (V, E)$ is given by a set V of *vertices* and a set E of *hyperedges*, where for each $e \in E$ we have $e \subseteq V$. A hypergraph is called k-uniform if $|e| = k$ for all $e \in E$. For a set X, let us denote $\binom{X}{k} = \{A \subseteq X : |A| = k\}$. Let us also denote $\mathcal{H}_{n,k} = \{H = ([n], E) : E \subseteq \binom{[n]}{k}\}$, i.e., the family of every k-uniform hypergraph with n vertices. Given n, k and t, we say that a hypergraph $H \in \mathcal{H}_{n,k}$ is *t-intersecting* if for any two hyperedges f, g of H we have $|f \cap g| \geq t$. A natural question about this problem would be finding, for n, k and t, a t-intersecting k-uniform hypergraph $H = ([n], E)$ that maximizes $|E|$. For $t = 1$ this is the well-known Erdős-Ko-Rado problem [5]. The authors of [5] determined that, for $n \geq 2k$ the largest 1-intersecting k-uniform hypergraph is isomorphic to $H = ([n], \mathcal{S})$, where $\mathcal{S} = \{F \in \binom{[n]}{k} : 1 \in F\}$. Ahlswede and Khachatrian [1] found the optimum t-intersecting k-uniform hypergraph for every n, k, t and, if n is large enough with respect to k and t, they determined that the optimum hypergraph is isomorphic to $\mathcal{S}(t) = \{F \in \binom{[n]}{k} : [t] \subseteq F\}$. A (q,t)-coloring of a hypergraph H is a function associating every hyperedge from $E(H)$ with a color in $[q]$ where any two hyperedges f and g with the same color must satisfy $|f \cap g| \geq t$. Let $Q_{q,t}(H)$ be the family of all (q,t)-colorings of the hypergraph H. Let us call $\kappa(H, q, t) = |Q_{q,t}(H)|$ by the number of (q,t)-colorings of the hypergraph H. Finally, let $\mathrm{KC}(n, k, q, t)$ be the maximum number of (q,t)-colorings in some k-uniform hypergraph with n vertices, i.e.,

$$\mathrm{KC}(n,k,q,t) = \max_{H \in \mathcal{H}_{n,k}} \kappa(H,q,t), \tag{1}$$

where the maximum runs over the family $\mathcal{H}_n$ of all k-uniform hypergraphs on n vertices.

Before we state the main result determined by Hoppen, Kohayakawa and Lefmann, let us introduce an important definition.

Definition 1.1. *Given positive integers $q, k \geq 2$, $1 \leq t < k$, $c = c(q) = \lceil \frac{q}{3} \rceil \geq 1$ and $n \geq \max\{k, ct\}$, let $C \subseteq \binom{[n]}{t}$ be a set of cardinality c. The (C,k)-*complete hypergraph *$\mathcal{F}_C(n,k) = \mathcal{F}_C$ has $[n]$ as its set of vertices and every k-set from $[n]$ which contains some element from C is a hyperedge of $\mathcal{F}_C$.*

Theorem 1.1. *Given positive integers k, q and t, there is $n_0 > 0$ such that for all $n > n_0$, the equality $\kappa(H,q,t) = \mathrm{KC}(n,k,q,t)$ implies the following:*

(a) If $q \in \{2,3\}$ or if $q \geq 5$ and $k \geq 2t-1$, then H is isomorphic to $\mathcal{F}_C$, where the sets from C are mutually disjoint.

(b) If $q = 4$, then $H = \mathcal{F}_C$ for $C = \{t_1, t_2\}$ with $|t_1 \cap t_2| = t - 1$.

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(c) If $q \geq 5$ and $k < 2t - 1$, then $H = \mathcal{F}_C$ for $C = \{t_1, t_2, \ldots, t_{c(q)}\}$, where $|t_i \cup t_j| > k$, for all $1 \leq i < j \leq c(q)$.

Remark. *In case (a), for $q = 2$ the hypergraph above is optimum, for every $n \geq k$.*

Note that, as in [2], for $q \in \{2, 3\}$ the optimum hypergraph is the one with the maximum number of hyperedges avoiding two hyperedges sharing less than t elements, found by Erdős, Ko and Rado [5]. Moreover, for $q \geq 4$, this hypergraph is far from being optimum. Also, note that in item (c), Theorem 1.1 does not say precisely which hypergraph is optimum, since there are many hypergraphs satisfying this property. The authors of [6] proposed a conjecture about additional properties satisfied by the optimum hypergraph in this case.

Conjecture 1.1 (HKL-Conjecture)**.** *If $q \geq 5$, k and t are positive integers with $t < k < 2t - 1$, then there is $n_0 > 0$, such that, for $n > n_0$, a hypergraph $H = \mathcal{F}_C$ which satisfies $\kappa(H, q, t) = \mathrm{KC}(n, k, q, t)$ also satisfy $|t_i \cup t_j| = k + 1$ for any disjoint $t_i, t_j \in C$.*

Note that, even if the HKL-Conjecture were true, we would not know precisely which hypergraphs are optimum, because we still have many non-isomorfic hypergraphs satisfying the HKL-Conjecture.

Theorem 1.2. *The HKL-Conjecture is true for $q \leq 9$.*

2 Overview

To prove Theorem 1.2 we start with a new hypergraph $\mathcal{F}_C$, where C satisfies item (c) in Theorem 1.1. We assume that C does not satisfy the HKL-Conjecture. We create a shifting function φ, adapted from [5], which associates $\mathcal{F}_C$ with a new hypergraph $\mathcal{F}_{\widehat{C}}$. Then, we prove that $\kappa(\mathcal{F}_C, q, t) < \kappa(\mathcal{F}_{\widehat{C}}, q, t)$ using another function R that associates colorings of $\mathcal{F}_C$ with colorings of $\mathcal{F}_{\widehat{C}}$ in an injective, but not surjective way. Unfortunately, our approach does not prove the HKL-conjecture in general, since our shifting function cannot be applied to arbitrary hypergraphs. We now describe the hypergraphs to which shifting can be applied.

Definition 2.1. *Given $n > k > t$ and a family $C = \{t_1, \ldots, t_c\} \subseteq \binom{[n]}{t}$ of sets with $|C| = c$, we say C is a* reducible cover *with respect to $\{t_u, t_v\}$ if C satisfies; (a) $\forall\, i, j \in [c]$, $|t_i \cup t_j| > k$, (b) $|t_u \cup t_v| > k + 1$, (c) There are $x \in t_u \setminus t_v$ and $y \in t_v \setminus t_u$ such that, for every $z \in [c] \setminus \{u, v\}$ one of the following holds; (i) $x, y \in t_z$, (ii) $x, y \notin t_z$, (iii) $x \in t_z, y \notin t_z$ and $|t_v \cup t_z| > k + 1$, (iv) $x \notin t_z, y \in t_z$ and $|t_u \cup t_z| > k + 1$.*

For simplicity, given a reducible cover, we shall always assume that $x \in t_1$ and $y \in t_2$.

Definition 2.2. *For integers $n \geq k \geq t \geq 1$, let $C = \{t_1, \ldots, t_c\} \subseteq \binom{[n]}{t}$ be a reducible cover. We define the shifting function for hypergraphs $\varphi_{xy} \colon \mathcal{F}_C \longrightarrow \mathcal{F}_{\widehat{C}}$ as*

$$\varphi_{xy}(e) = \begin{cases} e' = (e \setminus \{y\}) \cup \{x\}, & x \notin e, t_2 \subseteq e, \\ e, & \textit{otherwise}, \end{cases} \tag{2}$$

where $\widehat{C} = (C \setminus \{t_2\}) \cup \{t_2'\}$ and $t_2' = (t_2 \setminus \{y\}) \cup \{x\}$.

In general, when the reducible cover is clear from context, given a set u with $y \in u$ and $x \notin u$, we shall write $u' = (u \setminus \{y\}) \cup \{x\}$.

Lemma 2.1. *Let $n \geq k \geq t$ be integers and C be a reducible cover. Let $e \in \mathcal{F}_C$ be a hyperedge. Then we have:*

(a) $|e| = |\varphi_{xy}(e)|$.

(b) The function $\varphi_{xy} \colon \mathcal{F}_C \longrightarrow \mathcal{F}_{\widehat{C}}$ from Definition 2.2 is bijective.

(c) $|\mathcal{F}_C| = |\mathcal{F}_{\widehat{C}}|$.

3 The optimum hypergraph for $q \in \{5, 6\}$

In this section, we prove Theorem 1.3 for $q \in \{5, 6\}$, where we know that the optimum hypergraph has a cover of size two and arguments are simpler.

Proof. Consider $q \in \{5, 6\}$ and let C be a cover of size c that does not satisfy the HKL-Conjecture. By Theorem 1.1 with $q \in \{5, 6\}$ we have $c = 2$, say $C = \{t_1, t_2\}$. As $|t_1| = |t_2|$ but $t_1 \neq t_2$, there are $x \in t_1 \setminus t_2$ and $y \in t_2 \setminus t_1$, which implies that C is a reducible cover. Consider $\widehat{C} = \{t_1, t_2'\}$, where $t_2' = (t_2 \setminus \{y\}) \cup \{x\}$.

Claim 3.1. *Given two hyperedges $u, v \in \mathcal{F}_C$, with $|u \cap v| \geq t$, we have $|\varphi_{xy}(u) \cap \varphi_{xy}(v)| \geq t$.*

Proof. For a contradition, let us suppose that there are $u, v \in \mathcal{F}_C$, with $|u \cap v| \geq t$, but $|\varphi_{xy}(u) \cap \varphi_{xy}(v)| < t$. In this case, φ_{xy} must change only one of the hyperedges. Without loss of generality, we suppose $\varphi_{xy}(u) = u$ and $\varphi_{xy}(v) = v'$. Then we have $|u \cap v'| < t$. From $|u \cap v| \geq t$, we have $y \in u$ and $x \notin u$. Then, $t_2 \subseteq u$, which implies $\varphi_{xy}(u) = u'$, a contradiction. □

Now, for every (q, t)-coloring Δ of $\mathcal{F}_C$, we define $\mathcal{M}(\Delta)$ as the (q, t)-coloring of $\mathcal{F}_{\widehat{C}}$ which associate with each hyperedge $\varphi_{xy}(e)$ the color of e in Δ. For simplicity, we may think of the function $\mathcal{M}$ as "keeping" the color of every hyperedge from $\mathcal{F}_C$ to $\mathcal{F}_{\widehat{C}}$.

Note that, on the one hand, $\mathcal{M}$ is an injective function because φ_{xy} is an injective function. This implies that $|Q_{q,t}(\mathcal{F}_C)| \leq |Q_{q,t}(\mathcal{F}_{\widehat{C}})|$. On the other hand, we claim that we can choose a pair $\varphi_{xy}(u), \varphi_{xy}(v) \in \mathcal{F}_{\widehat{C}}$ such that $t_1 \subseteq \varphi_{xy}(u)$, $y \notin \varphi_{xy}(u)$, $t_2' \subseteq \varphi_{xy}(v)$, $y \notin \varphi_{xy}(v)$ and $|\varphi_{xy}(u) \cap \varphi_{xy}(v)| = t$. Consider the following coloring $\widehat{\Delta}$ of $\mathcal{F}_{\widehat{C}}$. We choose a color α to associate with those two hyperedges $\varphi_{xy}(u)$ and $\varphi_{xy}(v)$. For the other hyperedges, we use two other colors, say yellow for those ones covered by t_1 and green for those ones covered by t_2'. Note that this is a (q, t)-coloring of $\mathcal{F}_{\widehat{C}}$. The existence of a (q, t)-coloring Δ of $\mathcal{F}_C$ such that $\mathcal{M}(\Delta) = \widehat{\Delta}$ requires that u and v have the same color in Δ, but this is not possible, as $|\varphi_{xy}(u) \cap \varphi_{xy}(v)| = t$ implies $|u \cap v| = t - 1$. Therefore, $\mathcal{M}$ is not surjective. To see that our claim is true, the hyperedges $\varphi_{xy}(u)$ and $\varphi_{xy}(v)$ may be defined as follows. Put t_1 in $\varphi_{xy}(u)$, t_2' in $\varphi_{xy}(v)$, and then choose vertices from $t_1 \setminus t_2'$ to add in $\varphi_{xy}(v)$ and vertices from $t_2' \setminus t_1$ to add in $\varphi_{xy}(u)$, aiming for the desired intersection. After that, it is enough to choose vertices outside $t_1 \cup t_2'$ until the hyperedges have size k (taking the same vertices for both if we still do not have the desired intersection). □

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Then, for five or six colors we have $|Q_{q,t}(\mathcal{F}_C)| < |Q_{q,t}(\mathcal{F}_{\widehat{C}})|$. Besides, there is only one cover satisfying the HKL-Conjecture in this case, so that we know that there is a single optimum hypergraph up to isomorphism.

4 Proof of the HKL-Conjecture for $q \in \{7, 8, 9\}$

Now, we shall give an overview of the proof of the HKL-Conjecture when $q \in \{7, 8, 9\}$. In this case, we need to be more careful, because Claim 3.1 does not hold for $q \geq 7$. An example of this follows below.

Example 4.1. *Consider $n = 8$, $k = 4$, $t = 3$, $q = 9$, $t_1 = \{x, 1, 2\}$, $t_2 = \{y, 3, 4\}$, $t_3 = \{1, 3, 5\}$, $C = \{t_1, t_2, t_3\}$. Note that $k < 2t - 1$, $q \geq 5$ and, in fact, C is a reducible cover. Consider the following (q, t)-coloring Δ of $\mathcal{F}_C$ (the colors appear between brackets at the right side of the hyperedges).*

Table 1: A coloring that we cannot keep colors.

Colored hyperedges	Contain
$\{x, y, 1, 2\}[a], \{x, 1, 2, 3\}[a], \{x, 1, 2, 4\}[a], \{x, 1, 2, 5\}[a], \{x, 1, 2, 6\}[a]$	t_1
$\{x, y, 3, 4\}[b], \{y, 1, 3, 4\}[\alpha], \{y, 2, 3, 4\}[b], \{y, 3, 4, 5\}[b], \{y, 3, 4, 6\}[b]$	t_2
$\{x, 1, 3, 5\}[c], \{y, 1, 3, 5\}[\alpha], \{1, 2, 3, 5\}[c], \{1, 3, 4, 5\}[c], \{1, 3, 5, 6\}[c]$	t_3

Note that $f = \{y, 1, 3, 4\}$ and $g = \{y, 1, 3, 5\}$ have the same color (and satisfy $|f \cap g| \geq t$), but $|\varphi_{xy}(f) \cap \varphi_{xy}(g)| < t$. For that reason, we cannot just "keep" the color of each hyperedge from $\mathcal{F}_C$ to $\mathcal{F}_{\widehat{C}}$. So we create a function R, called recoloring function. *When two hyperedges f and g, with $t_2 \subseteq f$, $t_3 \subseteq g$ and $|f \cap g| = t$ cannot keep the same color because $|f' \cap g| = t - 1$, the function R switches the colors of g and g'.*

Table 2: The coloring after recoloring.

Colored hyperedges	Contain
$\{x, y, 1, 2\}[a], \{x, 1, 2, 3\}[a], \{x, 1, 2, 4\}[a], \{x, 1, 2, 5\}[a], \{x, 1, 2, 6\}[a]$	t_1
$\{x, y, 3, 4\}[b], \{x, 1, 3, 4\}[\alpha], \{x, 2, 3, 4\}[b], \{x, 3, 4, 5\}[b], \{x, 3, 4, 6\}[b]$	t'_2
$\{x, 1, 3, 5\}[\alpha], \{y, 1, 3, 5\}[c], \{1, 2, 3, 5\}[c], \{1, 3, 4, 5\}[c], \{1, 3, 5, 6\}[c]$	t_3

Note that $g = \{y, 1, 3, 5\}$ and $g' = \{x, 1, 3, 5\}$ switched colors because of the hyperedge $f = \{y, 1, 3, 4\}$, which is covered by t_2, and assume color α and intersection exactly $t = 3$ with $\{y, 1, 3, 5\}$.

As in the previous case we prove that the function R is injective but not surjective, which requires much more careful case analysis. With that we obtain $|Q_{q,t}(\mathcal{F}_C)| < |Q_{q,t}(\mathcal{F}_{\widehat{C}})|$. This leads to the following result.

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Lemma 4.1. *If $\mathcal{F}_C$ is a hypergraph where C is reducible, then $\mathcal{F}_C$ is not optimum.*

Now, to obtain the result for $q \in \{7, 8, 9\}$, it is enough to prove the following lemma.

Lemma 4.2. *If C is a cover such that $|C| = c = 3$, then C either satisfies the HKL-Conjecture, or is reducible.*

In fact, the following stronger result holds for $|C| = 3$.

Lemma 4.3. *Let n and t be positive integers, and let $t_1, t_2, t_3 \in \binom{[n]}{t}$ be distinct sets, where $|t_1 \cap t_2| \leq \min\{|t_1 \cap t_3|, |t_2 \cap t_3|\}$. Then at least one of the following cases is true.*

(A) There are elements x, y such that $x \in t_1 \setminus (t_2 \cup t_3)$ and $y \in t_2 \setminus (t_1 \cup t_3)$.

(B) There are elements x, y such that $x \in (t_1 \cap t_3) \setminus t_2$ and $y \in (t_2 \cap t_3) \setminus t_1$.

Proof. By contradiction, let us suppose both cases (A) and (B) are not true. Now we consider three cases. (i) $t_1 \setminus t_3 \subseteq t_2$. (ii) $(t_3 \setminus t_1) \cap t_2 = \emptyset$. (iii) $t_1 \setminus t_3 \not\subseteq t_2$ e $(t_3 \setminus t_1) \cap t_2 \neq \emptyset$.

In case (i), from $t_1 \neq t_2$, there is some element x such that $x \in (t_1 \cap t_3) \setminus t_2$. Since (B) is false, we have $(t_2 \cap t_3) \setminus t_1 = \emptyset$, which implies $t_2 \cap t_3 \subseteq t_1 \cap t_2 \cap t_3 \subseteq t_1 \cap t_2$. But $t_1 \setminus t_3 \neq \emptyset$ and $t_1 \setminus t_3 \subset t_2$, So that $t_1 \cap t_2 \cap t_3 \subset t_1 \cap t_2$, which implies $t_2 \cap t_3 \subset t_1 \cap t_2$, a contradiction with $|t_1 \cap t_2| \leq \min\{|t_1 \cap t_3|, |t_2 \cap t_3|\}$.

In case (ii) we have $(t_3 \setminus t_1) \cap t_2 = \emptyset$, which implies $t_2 \cap t_3 \subseteq t_1 \cap t_2 \cap t_3 \subseteq t_1 \cap t_2$, but from $|t_1 \cap t_2| \leq \min\{|t_1 \cap t_3|, |t_2 \cap t_3|\}$, we have $t_2 \cap t_3 = t_1 \cap t_2 \cap t_3 = t_1 \cap t_2$. On the other hand, we can write t_2 in a better way, $t_2 = (t_2 \cap t_1) \cup (t_2 \cap (t_3 \setminus t_1)) \cup (t_2 \setminus (t_1 \cup t_3))$, but from $t_2 \not\subseteq t_1$ and $(t_3 \setminus t_1) \cap t_2 = \emptyset$, we have $t_2 \setminus (t_1 \cup t_3) \neq \emptyset$, which implies, supposing that (A) is false, $t_1 \setminus (t_2 \cup t_3) = \emptyset$, which implies $t_1 = (t_1 \cap t_2) \cup (t_1 \cap t_3)$. But $t_2 \cap t_3 = t_1 \cap t_2$, then $t_1 = (t_2 \cap t_3) \cup (t_1 \cap t_3)$, which implies $t_1 \subseteq t_3$, a contradiction.

In case (iii) we have $t_1 \setminus t_3 \not\subseteq t_2$, so that there is an element $x \in t_1 \setminus (t_2 \cup t_3)$, from (A) we have $t_2 \setminus (t_1 \cup t_3) = \emptyset$, which implies $t_2 \subset t_1 \cup t_3$. Then we have $t_2 = (t_1 \cap t_2 \cap t_3) \cup (t_2 \cap (t_3 \setminus t_1)) \cup (t_2 \cap (t_1 \setminus t_3))$. But, from the hypothesis of this case, we have $(t_3 \setminus t_1) \cap t_2 \neq \emptyset$, which implies there is an element $y \in (t_3 \cap t_2) \setminus t_1$, and from (B), we have $(t_1 \cap t_3) \setminus t_2 = \emptyset$, which implies $t_1 \cap t_3 \subseteq t_2$, so that $t_1 \cap t_3 \subseteq t_1 \cap t_2$. But $|t_1 \cap t_2| \leq \min\{|t_1 \cap t_3|, |t_2 \cap t_3|\}$, so we have $t_1 \cap t_3 = t_1 \cap t_2$, which implies $t_2 \cap (t_1 \setminus t_3) = \emptyset$. Thus, we have $t_2 = (t_1 \cap t_2 \cap t_3) \cup (t_2 \cap (t_3 \setminus t_1))$, which implies $t_2 \subseteq t_3$, a contradiction. □

The combination of Lemma 4.1 and Lemma 4.2 implies that every optimum hypergraph must satisfy the HKL-Conjecture. To extend this to $q \geq 10$ we have two problems. First, we think that Lemma 4.1 is true for $q \geq 10$ but we are not aware of a proof. Second, for $q \geq 10$, there are covers that do not satisfy HKL-Conjecture but are not reducible, which we call irreducible. See Figure 1 for $k = 7$ and $t = 5$.

So, another technique is needed to prove the HKL-Conjecture for $q \geq 10$. Further note that, even if the HKL-Conjecture is true for every q, there are a lot of different configurations whose pairwise unions have size $k + 1$. Therefore, it would be interesting to investigate which of these configurations yield the largest number of (q, t)-colorings. See Figure 2 for $k = 4$ and $t = 3$.

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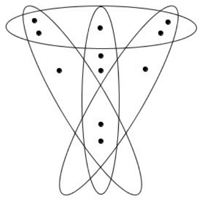

Figure 1: Irreducible cover of size 4 for $q = 12$, $k = 7$ and $t = 5$.

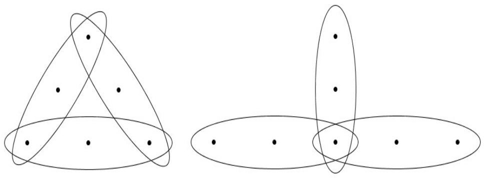

Figure 2: Distinct covers that satisfy the HKL-Conjecture for $q = 9$, $k = 4$ and $t = 3$.

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