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Eigenvalue Interlacing in Graphs

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Abstract. In this work we present eigenvalue interlacing results for the adjacency matrix A, the standard Laplacian matrix L, the normalized Laplacian matrix $\mathcal{L}$ and the signless Laplacian matrix Q associated with the edge subdivision operation. In addition, we show an eigenvalue interlacing result associated with the vertex contraction operation for the signless Laplacian matrix Q.

Keywords. Spectral graph theory, interlacing inequality, eigenvalues.

1 Introduction

Let $G = (V(G), E(G))$ be a *simple graph*, where $V(G) = \{v_1, v_2, ..., v_n\}$ is the *vertex set* and $E(G)$ is the *edge set*. Each edge in $E(G)$ can be represented by its ends such as vu. Denote the set of *neighbors* of v by $N_G(v)$ and the *degree* of v by $d_v = d_G(v) = |N_G(v)|$.

The *adjacency matrix* of G is $A(G) = (a_{ij})$, where $a_{ij} = 1$ if $v_iv_j \in E(G)$ and $a_{ij} = 0$ otherwise.

The *standard Laplacian* of G is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix with the (i,i)th entry having value d_{v_i}.

The *signless Laplacian* of G is defined as $Q(G) = D(G) + A(G)$.

The *normalized Laplacian* of G is $\mathcal{L}(G) = (\mathcal{L}_{ij})$ given by

$$\mathcal{L}_{ij} = \begin{cases} 1, \text{ if } v_i = v_j \text{ and } d_{v_i} \neq 0, \\ \dfrac{-1}{\sqrt{d_{v_i} d_{v_j}}}, \text{ if } v_iv_j \in E(G), \\ 0, \text{ otherwise.} \end{cases}$$

As it is well-known, the four matrices A, L, Q and $\mathcal{L}$ are all real symmetric. It follows that their eigenvalues are real numbers [2]. Denote the *spectrum* of an $n \times n$ real symmetric matrix M by $Spec(M) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, where we assume the eigenvalues $\lambda_i (i = 1, 2, \ldots, n)$ are arranged in nonincreasing order: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

In many papers, such as $[1, 3, 5]$, the interlacing relation of the spectra of matrix representations of graphs are studied. The eigenvalue interlacing provides a handy tool for obtaining inequalities and regularity results concerning the structure of graphs.

We first define the concept of the interlacing between real finite sequences.

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Definition 1.1 (Interlacing). *Let $S_1 : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $S_2 : \theta_1 \geq \theta_2 \geq \cdots \geq \theta_m$ be two real finite sequences with $m \leq n$. Set $a, b \in \mathbb{N}$, with $a + b \leq n - 1$ and $m \geq a + 1$, and let α, β be a nonnegative real numbers. S_2 is said to interlace with S_1 if*

$$\lambda_{i-a} + \alpha \geq \theta_i \geq \lambda_{i+b} - \beta \text{ for each } i = a+1, ..., min\{n-b, m\}.$$

Let $S_1 : Spec(Q(K_4)) = \{6, 2, 2, 2\}$ and $S_2 : Spec(Q(K_3)) = \{4, 1, 1\}$. Note that for $a = 0$, $b = 1$, $\alpha = 0$, and $\beta = 1$ we have

$$6 \geq 4 \geq 1, \quad 2 \geq 1 \geq 1 \quad \text{and} \quad 2 \geq 1 \geq 1.$$

This is an example of the eigenvalue interlacing result for Q associated with the vertex deletion operation.

Now we define some operations on graphs.

Definition 1.2 (Edge Subdivision). *The subdivision of an edge $e = vu \in E(G)$ produces a graph G_e, where $V(G_e) = V(G) \cup \{x_{vu}\}$, such that $x_{vu} \notin V(G)$, and $E(G_e) = (E(G) - e) \cup \{vx_{vu}, x_{vu}u\}$.*

Subdividing the edge vu in a graph G means that a new vertex x_{vu} is added to $V(G)$ and the edge vu is replaced in $E(G)$ by an edge vx_{vu} and an edge $x_{vu}u$.

Definition 1.3 (Vertex Contraction). *The contraction of a pair of vertices $v, u \in V(G)$ produces a graph $G_{\{v,u\}}$, where $V(G_{\{v,u\}}) = (V(G) - \{v, u\}) \cup \{x_{vu}\}$, x_{vu} being a new vertex with $N_{G_{\{v,u\}}}(x_{vu}) = [N_G(v) \cup N_G(u)] - \{v, u\}$, and $E(G_{\{v,u\}}) = [E(G) - (\{vz : z \in N_G(v)\} \cup \{uz : z \in N_G(u)\})] \cup \{x_{vu}z : z \in N_{G_{\{v,u\}}}(x_{vu})\}$.*

The contraction of a pair of vertices v and u in a graph G produces a graph in which the two vertices, v and u, are replaced with a single vertex x_{vu} such that x_{vu} is adjacent to the union of the vertices to which v and u were originally adjacent.

In this work, we will show that if a graph H can be obtained from G by an operation of edge subdivision, then the spectra of $A(H)$ and $A(G)$, $L(H)$ and $L(G)$, $Q(H)$ and $Q(G)$, $\mathcal{L}(H)$ and $\mathcal{L}(G)$ were interlaced. Moreover, we show an eigenvalue interlacing result associated with the vertex contraction operation for the signless Laplacian matrix.

This work is organized as follows. In Section 2 we show know results of eigenvalue interlacing, in particular, for the A, L, Q and $\mathcal{L}$ associated with the operations of edge deletion and vertex deletion. In Section 3 we present an eigenvalue interlacing result associated with the vertex contraction operation for Q. In Section 4 we give the proof of eigenvalue interlacing results for the A, L, Q and $\mathcal{L}$ associated with the subdivision edge operation. Finally, in section 5 we present the results achieved.

2 Known Interlacing Results

In this section, we show some known results of eigenvalue interlacing which will be used in the proofs of our main results about the eigenvalue interlacing for the operations of edge subdivision and vertex contraction.

Initially, we define the operations of edge deletion and vertex deletion.

Definition 2.1 (Edge Deletion)**.** *The deletion of an edge $e \in E(G)$ produces a graph $G - e$, where $V(G - e) = V(G)$ and $E(G - e) = E(G) - \{e\}$.*

Definition 2.2 (Vertex Deletion)**.** *The deletion of an vertex $v \in V(G)$ produces a graph $G - v$, where $V(G - v) = V(G) - \{v\}$ and $E(G - v) = E(G) - \{vz : z \in N_G(v)\}$.*

In [1], Hall et al. show eigenvalue interlacing results for the A, L and $\mathcal{L}$ associated with the operations of edge deletion and vertex deletion. Moreover, they also exhibit eigenvalue interlacing results for A and $\mathcal{L}$ associated with the vertex contraction operation. We present in the following theorems these results for the adjacency matrix.

Theorem 2.1. *Let G be a graph and let $H = G - e$, where e is an edge of G. If*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \text{ and } \theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$$

are the eigenvalues of $A(G)$ and $A(H)$, respectively, then

$$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+1} \text{ for each } i = 2, 3, ..., n-1,$$

$\lambda_1 \geq \theta_2$ and $\theta_n \leq \lambda_{n-1}$.

Theorem 2.2. *Let G be a graph and let $H = G - v$, where v is a vertex of G. If*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \text{ and } \theta_1 \geq \theta_2 \geq \cdots \geq \theta_{n-1}$$

are the eigenvalues of $A(G)$ and $A(H)$, respectively, then

$$\lambda_i \geq \theta_i \geq \lambda_{i+1} \text{ for each } i = 1, 2, ..., n-1.$$

Theorem 2.3. *Let G be a graph and let u and v be two distinct vertices of G. Define $H = G_{\{u,v\}}$ and let*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \text{ and } \theta_1 \geq \theta_2 \geq \cdots \geq \theta_{n-1}$$

be the eigenvalues of $A(G)$ and $A(H)$, respectively. Then

$$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+2} \text{ for each } i = 2, 3, ..., n-2,$$

$\theta_1 \geq \lambda_3$ and $\lambda_{n-2} \geq \theta_{n-1}$. If we assume that $N_G(u) \cap (N_G(v) \cup \{v\}) = \emptyset$ then, depending on the sign of θ_i, the above inequalities can be strengthened in one of two ways. Let k be such that $\theta_k \geq 0$ and $\theta_{k+1} < 0$. Then

$$\theta_i \geq \lambda_{i+1} \text{ for each } i = 1, 2, ..., k$$

and

$$\lambda_i \geq \theta_i \text{ for each } i = k+1, k+2, ..., n-1.$$

In [5], Wang and Belardo show eigenvalue interlacing results for Q associated with the operations of edge deletion and vertex deletion. In [3], Liu et al. show eigenvalue interlacing result for L associated with the vertex contraction operation (although they did not use this specific terminology). These, and the others results for L and $\mathcal{L}$ are similar to those already presented.

Let λ_i, $i = 1, 2, ..., n$ be the eigenvalues associated with $A(G)$, $L(G)$, $Q(G)$ or $\mathcal{L}(G)$ and let θ_i be the eigenvalues associated with $A(H)$, $L(H)$, $Q(H)$ or $\mathcal{L}(H)$. The following table shows all interlacing results previously mentioned.

Table 1: Know Interlacing Results.

	$H = G - e$	$H = G - v$	$H = G_{\{u,v\}}$
A	$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+1}$	$\lambda_i \geq \theta_i \geq \lambda_{i+1}$	$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+2}$
L	$\lambda_i \geq \theta_i \geq \lambda_{i+1}$	$\lambda_i \geq \theta_i \geq \lambda_{i+r}$	$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+1}$
$\mathcal{L}$	$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+1}$	$\lambda_{i-r+1} \geq \theta_i \geq \lambda_{i+r}$	$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+1}$
Q	$\lambda_i \geq \theta_i \geq \lambda_{i+1}$	$\lambda_i \geq \theta_i \geq \lambda_{i+1} - 1$	

Note: $r = d_G(v)$

To state the Theorem 2.4, we introduce the notation $M[I, J]$ as follows.

Let $I, J \subseteq \{1, 2, ..., n\}$ and $\overline{I} = \{1, 2, ..., n\}\backslash I$ denote the complement of I in $\{1, 2, ..., n\}$. Then we use the following notation to denote the submatrix of $M = (m_{ij})$:

$$M[I, J] = (m_{ij} : i \in I, j \in J).$$

In addition, the sum of the entries of M is denoted by $\sigma(M) = \sum_i \sum_j m_{ij}$.

The next theorem provides an important interlacing relation to the spectra of real symmetric matrices, a generalized version is in [3].

Theorem 2.4. *Let M,N be two $n \times n$ real symmetric matrices and let*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \text{ and } \theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$$

be the eigenvalues of M and N, respectively. If there exists $I \subseteq \{1, 2, ..., n\}$ with $|I| = 2$, such that

(1) $(M - N)[\overline{I}, \overline{I}] = \mathbb{O}$,

(2) $(M - N)[I, I]$ *has* 0 *sum (i.e.,* $\sigma((M - N)[I, I]) = 0$,

(3) $(M - N)[\overline{I}, I]$ *has* 0 *row sums,*

then

$$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+1} \text{ for each } i = 2, 3, ..., n - 1,$$

$\theta_1 \geq \lambda_2$ *and* $\lambda_{n-1} \geq \theta_n$.

3 Eigenvalue interlacing for Vertex Contraction

In this section, we give the eigenvalue interlacing result associated with the vertex contraction operation for the signless Laplacian Matrix. The proof use the Theorem 2.4.

Theorem 3.1. *Let G be a graph and let u and v be two distinct vertices of G with $N_G(u) \cap (N_G(v) \cup \{v\}) = \emptyset$. Define $H = G_{\{u,v\}}$ and let*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \text{ and } \theta_1 \geq \theta_2 \geq \cdots \geq \theta_{n-1}$$

be the eigenvalues of $Q(G)$ and $Q(H)$, respectively. Then

$$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+1} \text{ for each } i = 2, 3, ..., n-1,\ \theta_1 \geq \lambda_2 \text{ and } \lambda_{n-1} \geq \theta_n.$$

Proof. We assume without loss of generality that $u = v_1$ and $v = v_2$ and consider the graph $\widehat{H}$ obtained by deleting every edge from v_2 and adding a corresponding edge to v_1. Moreover, let $\widehat{\theta}_1 \geq \widehat{\theta}_2 \geq \cdots \geq \widehat{\theta}_n$ be the eigenvalues of $Q(\widehat{H})$.

Let $I = \{1, 2\}$, we get

$$(Q(\widehat{H}) - Q(G))[I, I] = \begin{pmatrix} d_G(v) & 0 \\ 0 & -d_G(v) \end{pmatrix}.$$

Thus $\sigma((Q(\widehat{H}) - Q(G))[I, I]) = 0$.

As $Q(\widehat{H})[\overline{I}, \overline{I}] = Q(G)[\overline{I}, \overline{I}]$ we have $(Q(\widehat{H}) - Q(G))[\overline{I}, \overline{I}] = \mathbb{O}$.

Note that

$$Q(G)[\overline{I}, I] = \begin{cases} 1, \text{ if } j = 1 \text{ and } v_i \in N_G(v_1), \\ 1, \text{ if } j = 2 \text{ and } v_i \in N_G(v_2), \\ 0, \text{ otherwise} \end{cases}$$

and

$$Q(\widehat{H})[\overline{I}, I] = \begin{cases} 1, \text{ if } j = 1 \text{ and } v_i \in N_G(v_1), \\ 1, \text{ if } j = 1 \text{ and } v_i \in N_G(v_2), \\ 0, \text{ otherwise.} \end{cases}$$

Then

$$(Q(\widehat{H}) - Q(G))[\overline{I}, I] = \begin{cases} 1, \text{ if } j = 1 \text{ and } v_i \in N_G(v_2), \\ -1, \text{ if } j = 2 \text{ and } v_i \in N_G(v_2), \\ 0, \text{ otherwise.} \end{cases}$$

It follows that $(Q(\widehat{H}) - Q(G))[\overline{I}, I]$ has 0 row sums. By Theorem 2.4, we get

$$\lambda_{i-1} \geq \widehat{\theta}_i \geq \lambda_{i+1} \text{ for each } i = 2, 3, ..., n-1,$$

$\widehat{\theta}_1 \geq \lambda_2$ and $\lambda_{n-1} \geq \widehat{\theta}_n$.

Note that the graph $\widehat{H}$ differ from graph H only by has an additional vertex, v_2, of degree zero, then $Spec(Q(H)) = Spec(Q(\widehat{H})) \cup \{0\}$.

Thus we conclude that

$$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+1} \text{ for each } i = 2, 3, ..., n-1 \text{ and } \theta_1 \geq \lambda_2.$$

$\square$

4 Eigenvalue interlacing for Edge Subdivision

For the proof of eigenvalue interlacing results associated with the edge subdivision operation, we use the following lemma.

Lemma 4.1. *Let G be a graph and let $e = vu \in E(G)$, then $G - e = G_e - x_{vu}$, where $x_v u$ is the added vertex in the subdivision of the edge e.*

We are now ready to give the proof for the results of eigenvalue interlacing associated with the edge subdivision operation.

Theorem 4.1. *Let G be a graph and let $H = G_e$, where $e = vu$ is an edge of G. If*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \text{ and } \theta_1 \geq \theta_2 \geq \cdots \geq \theta_{n+1}$$

are the eigenvalues of $A(G)$ and $A(H)$, respectively, then

$$\lambda_{i-2} \geq \theta_i \geq \lambda_{i+1} \text{ for each } i = 3, 4, ..., n-1,$$

$\theta_1 \geq \lambda_2$, $\theta_2 \geq \lambda_3$, $\lambda_{n-2} \geq \theta_n$ and $\lambda_{n-1} \geq \theta_{n+1}$.

Proof. Let $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$ be the eigenvalues of $A(G - e)$.

By Theorem 2.1 we have

$$\lambda_{i-1} \geq \beta_i \geq \lambda_{i+1} \text{ for each } i = 2, 3, ..., n-1, \quad (1)$$

$\beta_1 \geq \lambda_2$ and $\lambda_{n-1} \geq \beta_n$.

By Theorem 2.2 and Lemma 4.1 we have

$$\theta_i \geq \beta_i \geq \theta_{i+1} \text{ for each } i = 1, 2, ..., n. \quad (2)$$

From (1) and (2), we get

$$\lambda_{i-2} \geq \theta_i \geq \lambda_{i+1} \text{ for each } i = 3, 4, ..., n-1,$$

$\theta_1 \geq \lambda_2$, $\theta_2 \geq \lambda_3$, $\lambda_{n-2} \geq \theta_n$ and $\lambda_{n-1} \geq \theta_{n+1}$. □

The theorems for matrices L, $\mathcal{L}$ and Q are similar and are shown in the following table. Their proofs uses the respective results presented in Table 1.

Table 2: Interlacing Results for Edge Subdivision.

	$H = G_e$
A	$\lambda_{i-2} \geq \theta_i \geq \lambda_{i+1}$ for each $i = 3, 4, ..., n-1$, $\theta_1 \geq \lambda_2$, $\theta_2 \geq \lambda_3$, $\lambda_{n-2} \geq \theta_n$, $\lambda_{n-1} \geq \theta_{n+1}$.
L	$\lambda_{i-2} \geq \theta_i \geq \lambda_{i+1}$ for each $i = 3, 4, ..., n-1$, $\theta_1 \geq \lambda_2$, $\theta_2 \geq \lambda_3$, $\lambda_{n-2} \geq \theta_n$, $\lambda_{n-1} \geq \theta_{n+1}$.
$\mathcal{L}$	$\lambda_{i-3} \geq \theta_i \geq \lambda_{i+2}$ for each $i = 1, 2, ..., n+1$, where $\lambda_i = 2$ for $i \leq 0$ and $\lambda_i = 2$ for $i \geq n+1$.
Q	$\lambda_{i-1} + 1 \geq \theta_i \geq \lambda_{i+1}$ for each $i = 2, 3, ..., n-1$, $\theta_1 \geq \lambda_2$ and $\lambda_{n-1} + 1 \geq \theta_n$.

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5 Conclusion

The results presented in this work complete table 1. Moreover, we can add a new column to this table with the results obtained from the edge subdivision operation. The following table shows the main eigenvalue interlacing results for the matrices A, L, $\mathcal{L}$ and Q associated with some operations on graphs.

Table 3: Interlacing Results.

	$H = G - e$	$H = G - v$	$H = G_{\{u,v\}}$	$H = G_e$
A	$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+1}$	$\lambda_i \geq \theta_i \geq \lambda_{i+1}$	$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+2}$	$\lambda_{i-2} \geq \theta_i \geq \lambda_{i+1}$
L	$\lambda_i \geq \theta_i \geq \lambda_{i+1}$	$\lambda_i \geq \theta_i \geq \lambda_{i+r}$	$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+1}$	$\lambda_{i-2} \geq \theta_i \geq \lambda_{i+1}$
$\mathcal{L}$	$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+1}$	$\lambda_{i-r+1} \geq \theta_i \geq \lambda_{i+r}$	$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+1}$	$\lambda_{i-3} \geq \theta_i \geq \lambda_{i+2}$
Q	$\lambda_i \geq \theta_i \geq \lambda_{i+1}$	$\lambda_i \geq \theta_i \geq \lambda_{i+1} - 1$	$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+1}$	$\lambda_{i-1} + 1 \geq \theta_i \geq \lambda_{i+1}$

Note: $r = d_G(v)$

The results for L, $\mathcal{L}$ and Q associated with the vertex contraction operation require that the contracted vertices are not adjacent or have common neighbors.

Note that the edge subdivision operation can be reversed by the contraction of two adjacent vertices, but the opposite relationship is not valid. The result associated with the edge subdivision operation is a complement for the case in which adjacent vertices are contracted and one of them has degree 2, because in this case the contraction of adjacent vertices is the inverse operation of edge subdivision.

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