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H (div) approximations based on *hp*-adaptive curved meshes using quarter point elements

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Abstract. **H**(div)-conforming finite element subspaces based on curved quadrilateral meshes, with *hp*-adaptation, are constructed to be applied in flux approximations of the mixed element formulation. In order to validate the implementation, a test problem with square-root singularity at a boundary point is simulated. The results demonstrate exponential rates of convergence, and a dramatic error reduction when quarter-point elements are applied close to the singularity. Using static condensation, the global condensed matrices to be solved have reduced dimension, which is proportional to the dimension of border fluxes.

Keywords. **H**(div) finite element space, *hp* adaptivity, curves elements

1 Introduction

Several methods have been developed for the construction **H**(div)-conforming approximations spaces to be applied in flux approximations of the mixed element formulation. In some contexts the vector basis functions are constructed on the master element and then they are transformed to the elements of the partition by Piola transformations, as described in [2, 9]. The constructions of hierarchical high order spaces in [5, 11, 12] are based on the properties of the De Rham complex.

Another methodology for the construction of **H**(div)-conforming approximation spaces is proposed in [10], which has been extended to affine *hp*-adaptive meshes formed by triangular or quadrilateral elements in [6], and to three-dimensional affine tetrahedral, hexahedral and prismatic meshes in [4]. The principle is to choose appropriate constant vector fields, based on the geometry of each element, which are multiplied by an available

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set of H^1 hierarchical scalar basic functions. The assembling of the vector basis functions is a direct consequence of the properties of the properly chosen vector fields, and of the continuity of the scalar basic functions.

For the applications of the present paper, $\mathbf{H}(\text{div})$-conforming finite element subspaces based on curved quadrilateral meshes, with *hp*-adaptation, are constructed. Therefore, vectorial shape functions are firstly constructed on the master element, and then they are mapped to the geometrical elements by the Piola transformation. The current interest is on quarter point elements, which are widely used in fracture mechanics in order to adapt their geometry to the behaviour of the solution close to a singular boundary point, which have been originally developed by [1, 8]. For this kind of curved elements, scalar shape functions inherit the singular behaviour of the geometric map, resulting in rational shape functions.

For the results of the present paper, the implementation of the $\mathbf{H}(\text{div})$ spaces using *hp*-meshes with quarter point elements, and their consistent applications to flux approximation on a mixed formulation, are performed in the NeoPZ [5] computational platform, which is an open-source object-oriented project providing a comprehensive set of high performance tools for finite element simulations, including *hp* adaptivity [3].

2 Approximation spaces in H(div,Ω)

Let Γ be a mesh on a domain $\Omega \subset \mathbb{R}^2$ formed by elements K. Approximation subspaces in

$$\mathbf{H}(\text{div}, \Omega) = \left\{ \mathbf{q} \in L^2(\Omega)^2; \nabla.\mathbf{q} \in L^2(\Omega) \right\},$$

which are defined piecewise over the elements of Γ, require that the local pieces $\mathbf{q}_K = \mathbf{q}|_K$ should be assembled by keeping continuous normal components across common element edges. The proposed methodology used for the construction of such kind of approximation subspaces follows a sequence of steps described below. We shall be concerned with quadrilateral meshes, without limitation on hanging sides and approximation order distribution $\mathbf{k} = (k_K)$.

1. To each element K, there is a geometric mapping $\mathbf{x} : \hat{K} \to K$, associating each point $\boldsymbol{\xi} \in \hat{K}$ of the rectangular master element $\hat{K}$ to a point $\mathbf{p} = \mathbf{x}(\boldsymbol{\xi}) \in K$. An isomorphism $\mathbb{F} : \hat{\varphi} \to \varphi$, mapping scalar functions of $H^1(\hat{K})$ to scalar functios of $H^1(K)$, is induced by the geometric mapping. It also induces a contravariant Piola transformation $\mathbb{F}^{div} : \hat{\mathbf{\Phi}} \to \mathbf{\Phi}$, an isomorphism mapping vector fields $\hat{\mathbf{\Phi}} \in \mathbf{H}(\text{div}, \hat{K})$ to vector fields $\mathbf{\Phi} \in H(\text{div}, K)$, which are defined in geometrical elements K by the formula $\mathbf{\Phi} = \mathbb{F}[\frac{1}{det\, \mathbf{J}} \mathbf{J} \hat{\mathbf{\Phi}}]$, where $\mathbf{J} = \nabla \mathbf{x}$ is the Jacobian of the geometric mapping.

2. A family of hierarchical bases $\mathbf{B}_{k_K}^{\hat{K}} = \{\hat{\mathbf{\Phi}}\}$ is given, where the parameter k_K refers to the total degree of the polynomials in $\mathcal{Q}_{k(K)}$ used in their definitions, as proposed in [10]. The principle is to choose appropriate constant vector fields $\hat{\mathbf{v}}$, based on

[5] http://github.com/labmec/neopz

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the geometry of the master element, which are multiplied by an available set of H^1 hierarchical scalar basic functions $\hat{\varphi}$ in order to get $\hat{\mathbf{\Phi}} = \hat{\varphi}\hat{\mathbf{v}}$.

3. A family of hierarchical vectorial bases $\mathbf{B}_{k_K}^K = \{\mathbf{\Phi}\}$ is defined over K by the Piola transformation $\mathbf{\Phi} = \mathbb{F}^{div}(\hat{\mathbf{\Phi}})$. There are shape functions of interior type, with vanishing normal components over all element edges. Otherwise, $\mathbf{\Phi}$ is classified as of edge type, and its normal component on the edge associated to it coincides with the restriction of the scalar shape function $\varphi = \mathbb{F}\hat{\varphi}$ used in its definition, and vanishes over the other edges.

4. Construction of approximation subspaces $\subset \mathbf{H}(\text{div}, \Omega)$ formed by functions $\mathbf{q} \in \left[L^2(\Omega)\right]^2$, which are defined piecewise over the elements of Γ by local functions $\mathbf{q}_K = \mathbf{q}|_K \in span\ \mathbf{B}_{k_K}^K \subset \mathbf{H}(\text{div}, K)$. They can easily be assembled to get continuous normal components on the elements interfaces. This property is obtained as a consequence of the particular properties verified by the proposed vectorial shape functions and of the continuity of the scalar shape functions used in their construction.

3 Application to mixed finite element formulation

Consider a model Poisson problem expressed as:

$$\begin{aligned} \boldsymbol{\sigma} &= -\nabla u \quad \text{in} \quad \Omega, &(1)\\ \nabla \cdot \boldsymbol{\sigma} &= f \quad \text{in} \quad \Omega, &(2)\\ u &= u_d \quad \text{on} \quad \Gamma_d, &(3)\\ \nabla u \cdot \boldsymbol{\eta} &= g \quad \text{on} \quad \Gamma_N, &(4)\end{aligned}$$

where $\Omega \subset \mathbb{R}^2$ is the computational domain with Lipchitz boundary $\partial\Omega = \Gamma_d \cup \Gamma_N$, $\Gamma_d \cup \Gamma_N = \emptyset$, where Γ_d and Γ_N denotes the Dirichlet and Neumann boundary condition respectively and $\boldsymbol{\eta}$ denotes the outward unit normal to boundary.

This problem can be expressed in the form: to find $u \in U = L^2(\Omega)$ and $\boldsymbol{\sigma} \in \mathbf{V} = \{\boldsymbol{q} \in H(div, \Omega) : \boldsymbol{q} \cdot \boldsymbol{\eta}|_{\Gamma_N} = -g\}$, such that $\forall \varphi \in U$ and $\forall \mathbf{q} \in H(div, \Omega)$, with $\boldsymbol{q} \cdot \boldsymbol{\eta}|_{\Gamma_N} = 0$,

$$\begin{aligned} a(\boldsymbol{\sigma}, \mathbf{q}) + b(\mathbf{q}, u) &= c(\mathbf{q}),\\ b(\boldsymbol{\sigma}, \varphi) &= \ell(\varphi) \end{aligned} \qquad (5)$$

where $a(\boldsymbol{\sigma}, \mathbf{q}) = \int_\Omega \boldsymbol{\sigma} \cdot \mathbf{q}\ d\Omega$, $b(\mathbf{q}, u) = -\int_\Omega u \nabla \cdot \mathbf{q}\ d\Omega$, $c(\mathbf{q}) = -\int_{\Gamma_d} u_d\, \mathbf{q} \cdot \boldsymbol{\eta}\, ds$ and $\ell(\varphi) = -\int_\Omega f\ \varphi\ d\Omega$. In typical $\mathbf{H}(\text{div})$-conforming discretized versions of the mixed formulation, approximate solutions $\boldsymbol{\sigma}_h$ and u_h are searched in finite dimensional subspaces $\mathbf{V}_h \subset \mathbf{V}$ and $U_h \subset U$.

In matrix form, the discrete version of system (5) can be written as

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma}_h \\ u_h \end{pmatrix} = \begin{pmatrix} -c_h \\ -f_h \end{pmatrix}, \qquad (6)$$

where the matrices A and B correspond to discrete versions of the bilinear forms $a(\boldsymbol{\sigma}, \mathbf{q})$ and $b(\boldsymbol{\sigma}, \varphi)$. The vectors $\boldsymbol{\sigma}_h$ and u_h denote the dual and primal degrees of freedom, c_h referes to the discrete duality term $\int_{\Gamma_d} u_d \, \mathbf{q} \cdot \boldsymbol{\eta} ds$ and f_h is associated to the L^2 projection of the forcing term on U_h.

4 Approximation spaces

Following the developments in [4], we shall consider two stable configuration cases for approximation spaces to be used for primal u and dual $\boldsymbol{\sigma}$ variables in discretized versions of the mixed formulation. In both cases, the primal variable is approximated in subspaces of $L^2(\Omega)$ formed by piecewise functions $u|_K = u_K = \mathbb{F}\hat{u}_K$, without any continuity constraint, as in typical discretized mixed formulations [2].

The first configuration considers u_K mapped from polynomials $\hat{u}_K \in \mathcal{Q}_{k_K}$, of variable maximum degree k_K. The dual variable $\boldsymbol{\sigma}$ is searched in approximation spaces $\subset \mathbf{H}(\mathrm{div}, \Omega)$ formed by vectorial functions $\mathbf{Q}$ such that $\boldsymbol{q}_K = \boldsymbol{q}|_K \in span \ \mathbf{B}^{K*}_{k_K}$, where the bases $\mathbf{B}^{K*}_{k_K} \subset \mathbf{B}^{K}_{k_K+1}$ are formed by enriching $\mathbf{B}^{K}_{k_K}$ with interior shape functions $\boldsymbol{\Phi} = \mathbb{F}^{div}(\hat{\boldsymbol{\Phi}})$, corresponding to $\hat{\boldsymbol{\Phi}} \in \mathbf{B}^{\hat{K}}_{k_K+1}$ whose divergence $\nabla \cdot \hat{\boldsymbol{\Phi}} \in \mathcal{Q}_{k_K}$. This is the type of RT_k approximations. The resulting set of approximations spaces is classified as being of $\mathbf{Q}^*_\mathbf{k} \, Q_\mathbf{k}$ type.

Another type of approximation configuration is classified as being of $\mathbf{Q}^{**}_\mathbf{k} \, Q_{\mathbf{k}+1}$ type, where the primal approximations u_K are mapped from polynomials $\hat{u}_K \in \mathcal{Q}_{k_K+1}$, and $\mathbf{Q}^{**}_k$ refers to vectorial approximation spaces spanned by bases $\mathbf{B}^{K**}_{k_K} \subset \mathbf{B}^{K*}_{k_K+1}$, where the edge functions are restricted to those ones of $\mathbf{Q}^*_\mathbf{k}$ type.

As explained in [4], when computing sufficiently smooth solutions using $\mathbf{Q}^*_\mathbf{k} \, Q_\mathbf{k}$ space configurations based on affine regular meshes, optimal convergence rates of identical approximation orders $k+1$ are obtained for primal and dual variables, as well for $\nabla \cdot \boldsymbol{\sigma}$. For the $\mathbf{Q}^{**}_\mathbf{k} \, Q_{\mathbf{k}+1}$ configuration, higher convergence rate of order $k+2$ is obtained for the primal variable. Furthermore, after static condensation is applied, the condensed systems to be solved only involves the flux edge terms and a constant value for u in each element, and thus they have the same dimension in both space configuration.

5 Test problem and adaptive *hp*-refinement process

A model problem is considered with $f = 0$, and $\Omega = [-0.5, 0.5] \times [0, 0.5]$, where the exact solution is the harmonic function given in polar coordinates by $u = 2^{1/4}\sqrt{r}\cos(\frac{\theta}{2})$, that presents a square root singularity at at the boundary point $(x, y) = (0, 0)$ $(r = 0)$, where there is a change from Dirichlet boundary condition $u = 0$, for $x < 0$, to Neumann condition $\nabla u \cdot \boldsymbol{\eta} = 0$, for $x > 0$. Elsewhere, Neumann boundary condition is taken.

This test problem has been used in [7] to compare the performance of different finite element formulations, including continuous, discontinuous, mixed and primal hybrid finite element methods. As expected, the application of *hp*-refinement improves considerably the performance of all methods.

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Quarter point elements are defined by an 8-node quadratic geometry element with two side nodes moved to one quarter of the side. By moving the two side nodes, the scalar shape functions inherit the singular behaviour of the geometric map and result in rational shape functions.

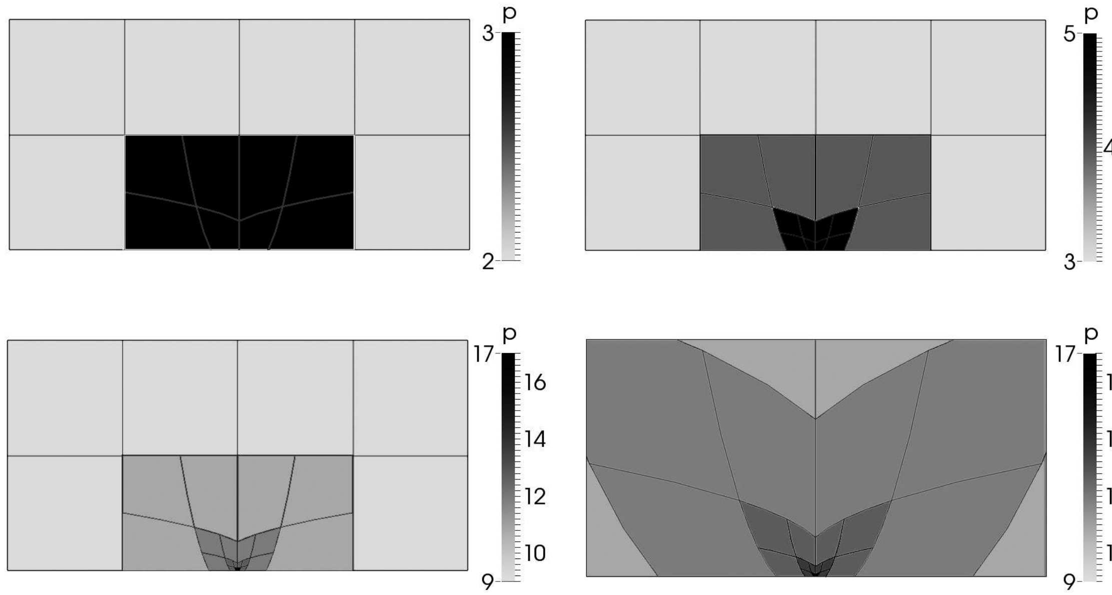

Figure 1: Illustration of the *hp* refinement process: initial mesh (top-left side), mesh after one refinement step (top-right side); mesh at the final refinement step (bottom-left side), and zoom in at the singular point (bottom-right side).

The construction of *hp*-adpative meshes with quarter-point elements is illustrated in Figure 1.

Our purpose is to use this kind of meshes for the simulation of the test problem by the mixed formulation using the space configurations of $\mathbf{Q}_\mathbf{k}^{*}\, Q_\mathbf{k}$ and $\mathbf{Q}_\mathbf{k}^{**}\, Q_{\mathbf{k}+1}$ types. Figure 2 shows the calculated L^2-norms of the dual and primal errors using these sequences of *hp*-adaptive curved meshes versus the number of equations solved after static condensation. For comparison, results for similar rectangular *hp*-meshes, without using quarter point elements, and for uniform meshes with constant $k = 2$ distribution are plotted. The results demonstrate exponential rates of convergence when using *hp*-adaptive meshes, and the dramatic error reduction when quarter-point meshes are applied. Furthermore, the accuracy in the primal variable improves when quarter point meshes and $\mathbf{Q}_\mathbf{k}^{**}\, Q_{\mathbf{k}+1}$ configuration are applied.

The effect of static condensation is also verified in terms of the size reduction of the global system to be solved, which is more significant with increasing order of approximation. At the finest levels of mesh refinement, the number of condensed equations amounts to more than 95% of the total number of equations, as shown in the Figure 3, which demonstrates the potential benefit of using $\mathbf{H}(\mathrm{div})$ approximation spaces in parallel computers.

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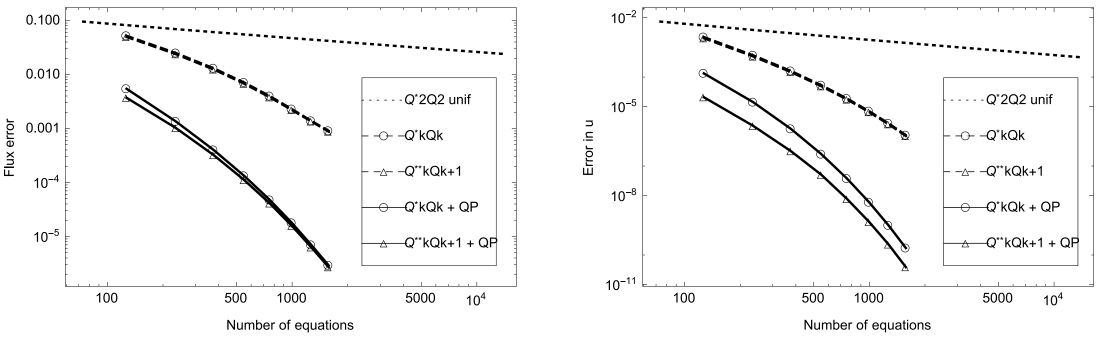

Figure 2: L^2-error in $\boldsymbol{\sigma}$ (left side), and in u (right side) using the mixed formulation based on $\mathbf{Q}^*_\mathbf{k}\ Q_\mathbf{k}$ (cicle), and $\mathbf{Q}^{**}_\mathbf{k}\ Q_{\mathbf{k}+1}$ (triangle) space configurations for *hp* curved meshes with quarter point elements (continuous lines), and similar *hp* rectangular meshes (dashed lines). The simple dotted curves correspond to simulations for uniform rectangular meshes with constant $k = 2$ distribution.

The CPU time necessary for computing the stiffness matrix of a quarter point element is the same as for any other H(div) element. As a standard procedure, our finite element library condenses the internal degrees of freedom of each element before assembling the global system of equations. The element matrices (and their static condensation) are performed in parallel, and the decomposition of the global system of equations is done serial. The NeoPZ finite element library also includes a frontal solver, where computation of element matrices and their decomposition is done simultaneously, in parallel.

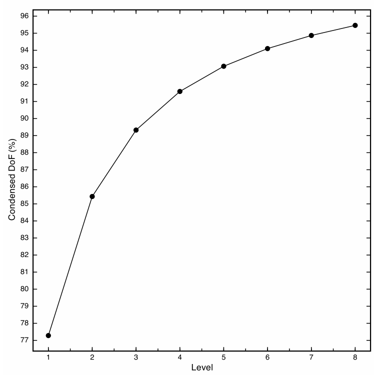

Figure 3: The effect of static condensation: condensed degrees of freedom (%) at each level of the *hp*-meshes.

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