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A Stabilized Hybrid-Mixed Finite Element Formulation for the Elasticity Problems

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Abstract. A stabilized hybrid dual-mixed finite element formulation is proposed to the elasticity problem in displacement and stress fields and a Lagrange multiplier identified *a priori* as the trace of the displacement field on the edges of the elements. The stabilization mechanisms, used to overcome the local compatibility condition (Ladyzhenskaya-Babuska-Brezzi condition), are activated by adding least squares residual forms of the governing equations in domain and on element boundary. Features of the formulation such as consistency, stability and local conservation are discussed. Numerical results for problems with smooth solution confirming optimal rates of convergence are presented.
Keywords. Elasticity, Hybridization, Mixed, Stabilization, Finite Elements.

1 Introduction

It is well-known that the mixed formulations are difficult to handle and to implement with their corresponding algebraic systems indefinite in general. To obtain enhanced stability without requiring much compatibility condition between spaces Loula *et al.* in [3] and Franca *et al.* in [2] proposed stabilized finite elements methods for structural mechanic such as Timoshenko beam and incompressible elasticity, so on. These type of stabilizations are derived by adding weighted residual forms of the governing equations to the mixed Galerkin formulation. In this case, are obtained the well-posedness of the formulation without compromising the flexibility in the choice of the approximation spaces.

Important contributions using discontinuous Galerkin methods have been presented for linear elasticity. We highlight the mixed formulations of Local Discontinuous Galerkin (LDG) developed by Cockburn, Schötzau and Wang [1]. A hybridized version of this formulation was presented by Soon, Cockburn and Stolarski in [5]. In Loula [4] is proposed a stabilized dual mixed-hybrid finite element method (SHDM-FEM) for Helmholtz problem. The stabilizing mechanisms are given by adding least squares residual of the governing

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equations in domain and on element boundary, but using discontinuous piecewise polynomials on a finite element partition. The continuity conditions are weakly enforced. In this hybrid method the only globally coupled degrees of freedom are those associated with the approximation of the displacement on boundary of the elements. The displacement and stress approximations in the interior of the elements are calculated using element-by-element post-processing.

An outline of the paper is organized as follows. In Section 2 we introduce the model problem of linear isotropic elasticity. In the Section 3 and 4 we present and discuss some properties of the stabilized and hybridized dual-mixed finite element formulation. Numerical experiments illustrating the convergence rate of the proposed method are shown in Section 5 and some concluding remarks are presented in Section 6.

2 Model Problem

Let Ω be a bounded open domain in $\mathbb{R}^{n_{sd}}$, $n_{sd} = 2, 3$, occupied by a deformable medium, with Lipschitz-continuous boundary $\Gamma = \partial\Omega$ and subject to external body force $\mathbf{f} : \Omega \to \mathbb{R}^{n_{sd}}$. The isotropic linear elasticity problem is given as following.

S Problem: Find the displacement $\mathbf{u}(\mathbf{x})$: $\Omega \longrightarrow \mathbb{R}^{n_{sd}}$ and the Cauchy stress tensor $\boldsymbol{\sigma}(\mathbf{x})$: $\Omega \longrightarrow \mathbb{R}^{n_{sd}\times n_{sd}}_{sym}$, $\forall \mathbf{x} \in \Omega$, such that

$$\begin{aligned} \operatorname{div}(\boldsymbol{\sigma}) + \mathbf{f} &= 0 \qquad \text{in } \Omega \quad \text{(equilibrium equation)}, && (1)\\ \mathbb{D}\boldsymbol{\sigma} &= \boldsymbol{\epsilon}(\mathbf{u}) \quad \text{in } \Omega \quad \text{(constitutive equation)}, && (2) \end{aligned}$$

where the small strain tensor is given as the symmetrical part of the displacement gradient, $\boldsymbol{\epsilon}(\mathbf{u}) = \nabla^S \mathbf{u}$, and the boundary conditions are

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \bar{\mathbf{g}}_D \qquad \forall\, \mathbf{x} \in \Gamma_D, \quad \text{(Dirichlet b. c.)} && (3)\\ (\boldsymbol{\sigma}\cdot\mathbf{n})(\mathbf{x}) &= \bar{\mathbf{g}}_N \qquad \forall\, \mathbf{x} \in \Gamma_N, \quad \text{(Neumann b. c.)} && (4) \end{aligned}$$

where $\mathbf{n}$ is the outward unit normal vector on Γ, with $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$, $\bar{\mathbf{g}}_D : \Omega \to \mathbb{R}^{n_{sd}}$ is the prescribed displacement vector on Γ_D and $\bar{\mathbf{g}}_N : \Omega \to \mathbb{R}^{n_{sd}}$ is the prescribed surface traction vector on Γ_N. The isotropic fourth-order constitutive tensors, so-called of stiffness tensor $\mathbb{C}$ and compliance tensor $\mathbb{D} = \mathbb{C}^{-1}$, are given by

$$\mathbb{D}\boldsymbol{\sigma} = \frac{1}{2\mu}\Big(\boldsymbol{\sigma} - \frac{\lambda}{2\mu + n_{sd}\lambda}\operatorname{tr}(\boldsymbol{\sigma})\mathbb{I}\Big) \quad \text{and} \quad \mathbb{C}\boldsymbol{\epsilon}(\mathbf{u}) = \lambda \operatorname{div}(\mathbf{u})\mathbb{I} + 2\mu\boldsymbol{\epsilon}(\mathbf{u}), \qquad (5)$$

where μ and λ are Lamé constants of the material, with $\mu > 0$ and $\lambda + \frac{2\mu}{3} \geq 0$ owing to thermodynamic considerations, and $\operatorname{tr}(\boldsymbol{\sigma}) = \sum_{i=1}^{n_{sd}} \sigma_{ii}$. The Lamé coefficients are related to the *Poisson's ratio* ν and the *Young's modulus* E by

$$\lambda = \frac{\nu E}{(1+\nu)\,(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)}. \qquad (6)$$

In the compressible and nearly incompressible cases we assume that $\mathbb{C}$ and $\mathbb{D}$ are positive-definite tensors such that $C_{min}\boldsymbol{\phi}^T\boldsymbol{\phi} \leq \boldsymbol{\phi}^T\mathbb{C}\boldsymbol{\phi} \leq C_{max}\boldsymbol{\phi}^T\boldsymbol{\phi}$, $\forall \boldsymbol{\phi} \in \mathbb{R}^{n_{sd}\times n_{sd}}_{sym}$ and $D_{min}\boldsymbol{\phi}^T\boldsymbol{\phi} \leq \boldsymbol{\phi}^T\mathbb{D}\boldsymbol{\phi} \leq D_{max}\boldsymbol{\phi}^T\boldsymbol{\phi}$, $\forall \boldsymbol{\phi} \in \mathbb{R}^{n_{sd}\times n_{sd}}_{sym}$, where $C_{min}, C_{max}, D_{min}$ and D_{max} are constants such that $0 < C_{min} < C_{max}$ and $0 < D_{min} < D_{max}$.

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3 Stabilized Hybrid Dual-Mixed Formulation for Elasticity

Let Ω be a bounded open polygonal domain in $\mathbb{R}^{n_{sd}}$ with the boundary Γ. We denote by $\{\mathcal{T}_h\}$ a family of shape-regular partition of Ω such that $\mathcal{T}_h = \{\mathcal{K}\}_{e=1}^{Ne}$ is the union of all N_e finite elements $\mathcal{K}$ of the domain Ω. The partition $\mathcal{T}_h$ is indexed with the mesh parameter $h := \max_{\mathcal{K}\in\mathcal{T}_h} h_e$, where h_e is the *diameter* of element $\mathcal{K}$. Similarly, we define $\mathcal{E}_h$ as the collection of all the faces (or edges) of the partition $\mathcal{T}_h$. $\mathcal{E}_h = \{\partial\mathcal{K}\}$ is the union of all faces (or edges) of the elements $\mathcal{K} \in \mathcal{T}_h$. $\mathcal{E}_h^i$ is the set of interior faces (or edges) and $\mathcal{E}_h^\partial = \mathcal{E}_h \cap \partial\Omega$ is the set of boundary faces, such that $\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^\partial$.

The broken space of vector-valued functions $\boldsymbol{H}^1(\mathcal{T}_h)$ is defined on $\mathcal{T}_h$ such that $\boldsymbol{H}^1(\mathcal{T}_h) = \{\mathbf{v} \in [L^2(\Omega)]^{n_{sd}}; v_i|_{\mathcal{K}} \in L^2(\mathcal{K}), \nabla v_i \in L^2(\mathcal{K}), \forall\mathcal{K} \in \mathcal{T}_h, i = 1,..,n_{sd}\}$, being $L^2(\mathcal{K})$ the Lebesgue space in $\mathcal{K}$. The broken space of second-order tensor-valued functions $\boldsymbol{H}(\mathrm{div},\mathcal{T}_h)$ is given by $\boldsymbol{H}(\mathrm{div},\mathcal{T}_h) = \{\boldsymbol{\tau} \in [L^2(\Omega)]^{n\times n}; \mathrm{div}(\boldsymbol{\tau})|_{\mathcal{K}} \in [L^2(\mathcal{K})]^{n_{sd}}, \quad \forall\mathcal{K} \in \mathcal{T}_h\}$. In addition, $\boldsymbol{H}_D^{1/2}(\mathcal{E}_h) = \{\hat{\mathbf{v}} : \mathcal{E}_h \to \mathbb{R}^{n_{sd}}; \exists\mathbf{v} \in [H_D^1(\Omega)]^{n_{sd}}, \hat{\mathbf{v}} = \gamma_h\mathbf{v}\}$ is the Lagrange multiplier space, where $H_D^1(\Omega) = \{v \in L^2(\Omega); \nabla v \in [L^2(\Omega)]^{n_{sd}};\ \gamma v = 0 \text{ on } \Gamma_D\}$ and $\gamma\phi = \phi|_\Gamma$ is the trace of the function $\phi \in H^1(\Omega)$ on the boundary Γ; and $\gamma_h\phi = \phi|_{\mathcal{E}_h}$ is the trace of function $\phi \in H^1(\Omega)$ on the structure of faces $\mathcal{E}_h$. The inner product and associated norm to the Lebesgue space on the partition $\mathcal{T}_h$ are $(\mathbf{u},\mathbf{v})_{\mathcal{T}_h} = \sum_{\mathcal{K}\in\mathcal{T}_h}(\mathbf{u},\mathbf{v})_{\mathcal{K}} = \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} \mathbf{u}\cdot\mathbf{v}\, d\mathcal{K}$ and $\|\mathbf{v}\|_{0,\mathcal{T}_h} = \{\sum_{\mathcal{K}\in\mathcal{T}_h}\|\mathbf{v}\|_{0,\mathcal{K}}^2\}^{1/2} = [\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} \mathbf{u}\cdot\mathbf{v}\ d\mathcal{K}]^{1/2}$. The associated norms with the spaces $\boldsymbol{H}_D^{1/2}(\mathcal{E}_h)$ and $\boldsymbol{H}(\mathrm{div},\mathcal{T}_h)$ are $\|\hat{\mathbf{v}}\|_{1/2,\mathcal{E}_h} = \inf_{\mathbf{v}\in[H_D^1]^{n_{sd}}(\Omega)}\{|\mathbf{v}|_{1,\Omega};\ \gamma_h\,\mathbf{v} = \hat{\mathbf{v}} \text{ on } \mathcal{E}_h\}$ and $\|\boldsymbol{\tau}\|_{\mathrm{div},\mathcal{T}_h} = \{\sum_{\mathcal{K}\in\mathcal{T}_h}\|\boldsymbol{\tau}\|_{\mathrm{div},\mathcal{K}}^2\}^{1/2} = \{\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}}(\tau : \tau + \mathrm{div}\tau\cdot\mathrm{div}\tau)\ d\mathcal{K}\}^{1/2}$, respectively.

Given the elements $\mathcal{K}$ and $\mathcal{K}'$, included in the partition $\mathcal{T}_h$, which share the face $e \in \mathcal{E}_h^i$ such that $e = \partial\mathcal{K}\cap\partial\mathcal{K}'$, we defined the average $\{\!\{\cdot\}\!\}$ and jump operators $[\![\cdot]\!]$ to vector-valued functions $\mathbf{v}$ and tensor-valued functions $\boldsymbol{\tau}$ defined on $\mathcal{T}_h$ as following.

$$\{\!\{\mathbf{v}\}\!\} = \frac{\mathbf{v}+\mathbf{v}'}{2} \quad \text{and} \quad \{\!\{\boldsymbol{\tau}\}\!\} = \frac{\boldsymbol{\tau}+\boldsymbol{\tau}'}{2} \qquad \text{on} \quad e \in \mathcal{E}_h^i, \tag{7}$$

$$[\![\mathbf{v}]\!] = \mathbf{v}\cdot\mathbf{n} + \mathbf{v}'\cdot\mathbf{n}' \qquad \text{and} \quad [\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau}\,\mathbf{n} + \boldsymbol{\tau}'\,\mathbf{n}' \quad \text{on} \quad e \in \mathcal{E}_h^i, \tag{8}$$

and for boundary faces $\{\!\{\mathbf{v}\}\!\} = \mathbf{v}$ and $\{\!\{\boldsymbol{\tau}\}\!\} = \boldsymbol{\tau}$ on $\partial\mathcal{K} \in \mathcal{E}_h^\partial$, and $[\![\mathbf{v}]\!] = \mathbf{v}\cdot\mathbf{n}$ and $[\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau}\,\mathbf{n}$ on $\partial\mathcal{K} \in \mathcal{E}_h^\partial$.

3.1 SHDM-Variational Formulation

In this section we develop a unconditionally stable hybrid-mixed formulation for the elasticity problem. We begin defining a given regular partition $\mathcal{T}_h$ of Ω and the product space $\mathcal{X}$ of the composite elements $\boldsymbol{\psi} = \{\boldsymbol{\tau},\mathbf{v},\hat{\mathbf{v}}\}$ given by

$$\mathcal{X} \quad = \quad \boldsymbol{H}(\mathrm{div},\mathcal{T}_h)\times\boldsymbol{H}^1(\mathcal{T}_h)\times\boldsymbol{H}_D^{1/2}(\mathcal{E}_h), \tag{9}$$

together with the norm $\|\boldsymbol{\psi}\|_{\mathcal{X}}$ defined by

$$\|\boldsymbol{\psi}\|_{\mathcal{X}} \quad = \quad \left(\|\boldsymbol{\tau}\|_{\mathrm{div},\mathcal{T}_h}^2 + \|\boldsymbol{\epsilon}(\mathbf{v})\|_{0,\mathcal{T}_h}^2 + \|h^{-1/2}(\hat{\mathbf{v}} - \{\!\{\mathbf{v}\}\!\})\|_{1/2,\mathcal{E}_h}^2 + \|h^{-1/2}[\![\mathbf{v}]\!]\|_{1/2,\mathcal{E}_h}^2\right)^{1/2} \tag{10}$$

Let $\mathbf{f} \in [L^2(\Omega)]^{n_{sd}}$ be the body force and $\bar{\mathbf{g}}_D = 0$ on Γ_D. For convenient choices of the parameters δ_1, δ_2 and β the stabilized dual mixed-hybrid formulation is stated as follows:

SHDM Problem: Find $\{\boldsymbol{\sigma}, \mathbf{u}, \hat{\mathbf{u}}\} \in \mathcal{X}$ for all $\{\boldsymbol{\tau}, \mathbf{v}, \hat{\mathbf{v}}\} \in \mathcal{X}$, such that

$$\begin{aligned} \sum_{\mathcal{K}\in\mathcal{T}_h} \Big\{ (\mathbb{D}\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{K}} + (\mathbf{u}, \operatorname{div}\boldsymbol{\tau})_{\mathcal{K}} - (\hat{\mathbf{u}}, \boldsymbol{\tau}\cdot\mathbf{n})_{\partial\mathcal{K}} + (\operatorname{div}\boldsymbol{\sigma} + \mathbf{f}, \mathbf{v})_{\mathcal{K}} & \quad + \\ \delta_1 \left(|\mathbb{D}|(\operatorname{div}\boldsymbol{\sigma} + \mathbf{f}), \operatorname{div}\boldsymbol{\tau}\right)_{\mathcal{K}} + \delta_2 \left(\boldsymbol{\sigma} - \mathbb{C}\nabla^S\mathbf{u}, \mathbb{D}\boldsymbol{\tau} - \nabla^S\mathbf{v}\right)_{\mathcal{K}} & \quad - \\ \beta\left(|\mathbb{C}|(\hat{\mathbf{u}} - \mathbf{u}),\ \mathbf{v}\right)_{\partial\mathcal{K}} \Big\} & = 0 \end{aligned} \qquad (11)$$

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \Big\{ \beta\left(|\mathbb{C}|(\hat{\mathbf{u}} - \mathbf{u}), \hat{\mathbf{v}}\right)_{\partial\mathcal{K}} - (\boldsymbol{\sigma}\mathbf{n}_{\mathcal{K}}, \hat{\mathbf{v}})_{\partial\mathcal{K}} + (\bar{\mathbf{g}}_N, \hat{\mathbf{v}})_{\partial\mathcal{K}\cap\Gamma_N} \Big\} = 0, \qquad (12)$$

where $\mathbf{n}_{\mathcal{K}}$ is the outward unit normal vector on $\partial\mathcal{K}$.

The stabilization parameters δ_1, δ_2 and β can be, in general, dependent of the mesh size h and are chosen to provide the best properties of stability and accuracy to the formulation. The terms in δ_1 appearing in formulation above activate the least-squares residual of the equilibrium equation and the terms in δ_2 do the same for the constitutive equation. Both terms contribute to improve the stability of the stress and displacement fields in the $\boldsymbol{H}(\operatorname{div}, \mathcal{T}_h)$ and $[H^1(\mathcal{T}_h)]^{n_{sd}}$ spaces, respectively. The terms in β enhance the stability of the Lagrange multipliers. The two last terms in the equation 12 weakly enforce the continuity of the traction vector on structure of faces $\mathcal{E}_h$, including the boundary Γ_N. The boundary condition on Γ_N is also imposed weakly. The parameters $|\mathbb{D}|$ and $|\mathbb{C}|$ are introduced in equation to adjust adequately the dimensions of the additional terms.

4 SHDM-Finite Element Approximation

For the sake of simplicity we assume that the domain $\Omega \subset \mathbb{R}^2$ is polygonal. Defining the finite element spaces such that

$$\mathcal{X}_h = \mathcal{W}_h \times \mathcal{V}_h \times \mathcal{M}_h, \qquad (13)$$

with

$$\mathcal{W}_h = \{\boldsymbol{\tau}_h \in \boldsymbol{H}(\operatorname{div}, \mathcal{T}_h);\ \tau_{ij} \in D_k(\mathcal{K}), \forall\mathcal{K} \in \mathcal{T}_h,\ i,j = 1...2\}, \qquad (14)$$

$$\mathcal{V}_h = \{\mathbf{v}_h \in \boldsymbol{H}^1(\mathcal{T}_h);\ v_i \in D_l(\mathcal{K}), \forall\mathcal{K} \in \mathcal{T}_h,\ i = 1...2\}, \qquad (15)$$

$$\mathcal{M}_h = \{\hat{\mathbf{v}}_h \in \boldsymbol{H}_D^{1/2}(\mathcal{E}_h);\ \hat{v}_i \in P_m(e), \forall e \in \mathcal{E}_h,\ i = 1...2\}, \qquad (16)$$

where $D_k(\mathcal{K}) = P_k(\mathcal{K})$, the space of polynomial functions of degree at most k in both variables, or $D_k(\mathcal{K}) = Q_k(\mathcal{K})$, the space of polynomial functions of degree at most k in each variable, $P_m(e)$ is the space of polynomial functions of degree at most m. Adding and rearranging the terms of the equations 11 and 12, and considering the parameters given by $\delta_1 > 0, \delta_2 = -\frac{1}{2}$ and $\beta = -\beta_0 \cdot h^{-1}$, $\beta_o > 0$, with $|\mathbb{C}|$ and $|\mathbb{D}|$ combined with β_o and δ_1, respectively, we can rewrite the formulation as follows.

SHDM-FEM: Let $\mathbf{f} \in [L^2(\Omega)]^{n_{sd}}$ be the body force and $\bar{\mathbf{g}}_D = 0$ on Γ_D. Find $\{\boldsymbol{\sigma}_h, \mathbf{u}_h, \hat{\mathbf{u}}_h\} \in \mathcal{X}_h$ such that

$$\boldsymbol{B}(\{\boldsymbol{\sigma}_h, \mathbf{u}_h, \hat{\mathbf{u}}_h\}, \{\boldsymbol{\tau}_h, \mathbf{v}_h, \hat{\mathbf{v}}_h\}) = \boldsymbol{F}(\{\boldsymbol{\tau}_h, \mathbf{v}_h, \hat{\mathbf{v}}_h\}) \quad \forall \{\boldsymbol{\tau}_h, \mathbf{v}_h, \hat{\mathbf{v}}_h\} \in \mathcal{X}_h, \tag{17}$$

with

$$\begin{aligned}\boldsymbol{B}(\{\boldsymbol{\sigma}_h, \mathbf{u}_h, \hat{\mathbf{u}}_h\}, \{\boldsymbol{\tau}_h, \mathbf{v}_h, \hat{\mathbf{v}}_h\}) = \sum_{\mathcal{K}\in\mathcal{T}_h} \Big\{ \frac{1}{2}(\mathbb{D}\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h)_{\mathcal{K}} + \delta_1 \,(\operatorname{div}\boldsymbol{\sigma}_h, \operatorname{div}\boldsymbol{\tau}_h)_{\mathcal{K}} \quad - \\ \frac{1}{2}\,(\mathbb{C}\nabla^S \mathbf{u}_h, \nabla^S \mathbf{v}_h)_{\mathcal{K}} \quad + \\ (\mathbf{u}_h, \operatorname{div}\boldsymbol{\tau}_h)_{\mathcal{K}} + (\operatorname{div}\boldsymbol{\sigma}_h, \mathbf{v}_h)_{\mathcal{K}} + \frac{1}{2}\,(\boldsymbol{\sigma}_h, \nabla^S \mathbf{v}_h)_{\mathcal{K}} + \frac{1}{2}\,(\nabla^S \mathbf{u}_h, \boldsymbol{\tau}_h)_{\mathcal{K}} \quad - \\ \beta_o\, h^{-1}\,((\hat{\mathbf{u}}_h - \mathbf{u}_h),\ (\hat{\mathbf{v}}_h - \mathbf{v}_h))_{\partial\mathcal{K}} - (\hat{\mathbf{u}}_h, \boldsymbol{\tau}_h \cdot \mathbf{n})_{\partial\mathcal{K}} - (\boldsymbol{\sigma}_h \cdot \mathbf{n}, \hat{\mathbf{v}}_h)_{\partial\mathcal{K}} \Big\},\end{aligned} \tag{18}$$

$$\boldsymbol{F}(\{\boldsymbol{\tau}_h, \mathbf{v}_h, \hat{\mathbf{v}}_h\}) = \sum_{\mathcal{K}\in\mathcal{T}_h} \Big\{ -(\mathbf{f}, \mathbf{v}_h)_{\mathcal{K}} - \delta_1\,(\mathbf{f}, \operatorname{div}\boldsymbol{\tau}_h)_{\mathcal{K}} - (\bar{\mathbf{g}}_N, \hat{\mathbf{v}}_h)_{\partial\mathcal{K}\cap\Gamma_N} \Big\}. \tag{19}$$

We observe that the approximation setting of the SHDM-formulation is non-conformal method, given important properties similar to those of the discontinuous Galerkin method. The following we are going to present some features of the SHDM-formulation.

4.1 Consistency

Let $\{\boldsymbol{\sigma}, \mathbf{u}\} \in \boldsymbol{H}(\operatorname{div}, \Omega) \times \boldsymbol{H}^1(\Omega)$ be the solution of the **problem S**, $\hat{\mathbf{u}} \in \boldsymbol{H}_D^{1/2}(\mathcal{E}_h)$ and $\{\boldsymbol{\tau}_h, \mathbf{v}_h, \hat{\mathbf{v}}_h\} \in \mathcal{X}_h$. Replacing the solution $\{\boldsymbol{\sigma}, \mathbf{u}\}$ in the equation 18, integrating by parts the term $(\mathbf{u}, \operatorname{div}\boldsymbol{\tau}_h)_{\mathcal{K}}$, considering that the Lagrange multiplier is given by $\hat{\mathbf{u}} = \{\!\{\gamma_h \mathbf{u}\}\!\} + \frac{1}{2\beta}\, [\![\gamma_h \boldsymbol{\sigma}]\!]$ on $\mathcal{E}_h$ and noting that the traction vector $\boldsymbol{\sigma}\mathbf{n}$ and the displacement $\mathbf{u}$ are continuous on structure of faces $\mathcal{E}_h$ we obtain

$$\boldsymbol{B}(\{\boldsymbol{\sigma}, \mathbf{u}, \hat{\mathbf{u}}\}, \{\boldsymbol{\tau}_h, \mathbf{v}_h, \hat{\mathbf{v}}_h\}) = \boldsymbol{F}(\{\boldsymbol{\tau}_h, \mathbf{v}_h, \hat{\mathbf{v}}_h\}) \quad \forall \{\boldsymbol{\tau}_h, \mathbf{v}_h, \hat{\mathbf{v}}_h\} \in \mathcal{X}, \tag{20}$$

what we conclude that the solution of the **problem S** is also solution of the SDMH-Formulation given by equation 17.

4.2 Local Conservation

Let $\mathcal{K}$ an element belonging to the interior of the domain partition $\mathcal{T}_h$ and choosing at the equation 17 the weighting functions $\mathbf{v}_h = \mathbf{1}$, $\hat{\mathbf{v}}_h = \mathbf{0}$ and $\boldsymbol{\tau}_h = \mathbf{0}$ in the element $\mathcal{K}$ and $\{\boldsymbol{\tau}_h, \mathbf{v}_h, \hat{\mathbf{v}}_h\} = \{\mathbf{0}, \mathbf{0}, \mathbf{0}\}$ elsewhere we obtain

$$\int_{\mathcal{K}} (\operatorname{div}(\boldsymbol{\sigma}_h) + \mathbf{f})\; d\mathcal{K} - \beta(h) \int_{\partial\mathcal{K}} (\hat{\mathbf{u}}_h - \mathbf{u}_h)\; ds = 0. \tag{21}$$

Exact local conservation (equilibrium) exists when the stability parameter β is null. We can show that under certain conditions the SHDM-formulation is also stable when $\beta = 0$ and therefore locally conservative.

4.3 Stability

Choosing $\boldsymbol{\psi}_h = \{\bar{\boldsymbol{\tau}}_h, \bar{\mathbf{v}}_h, \bar{\hat{\mathbf{v}}}_h\} = \{\boldsymbol{\sigma}_h, -\mathbf{u}_h, -\hat{\mathbf{u}}_h\}$ we prove

$$\sup_{\boldsymbol{\psi}_h \in \mathcal{X}_h} \frac{\boldsymbol{B}(\boldsymbol{\xi}_h, \boldsymbol{\psi}_h)}{\|\boldsymbol{\psi}_h\|_{\mathcal{X}_h}} \geq \frac{\boldsymbol{B}(\{\boldsymbol{\sigma}, \mathbf{u}, \hat{\mathbf{u}}\}, \{\bar{\boldsymbol{\tau}}_h, \bar{\mathbf{v}}_h, \bar{\hat{\mathbf{v}}}_h\})}{\|\{\bar{\boldsymbol{\tau}}_h, \bar{\mathbf{v}}_h, \bar{\hat{\mathbf{v}}}_h\}\|_{\mathcal{X}}} \geq \alpha \|\{\boldsymbol{\sigma}, \mathbf{u}, \hat{\mathbf{u}}\}\|_{\mathcal{X}},$$

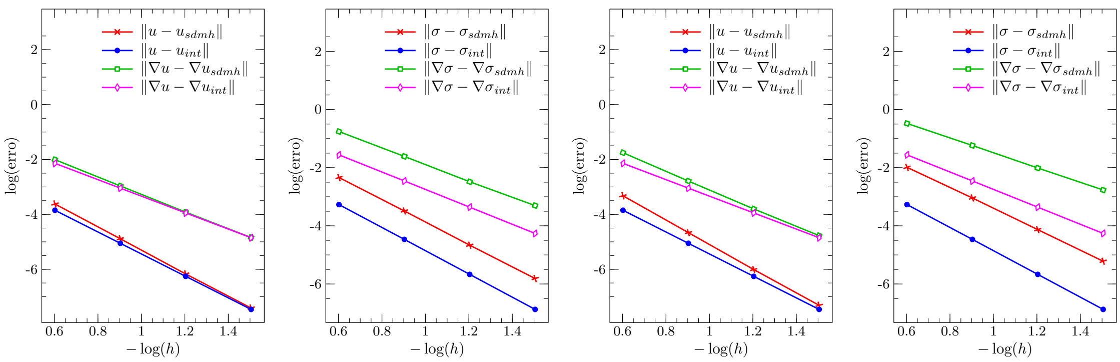
with the stability constant $\alpha = \min\{\frac{D_{min}}{2},\ \delta_1,\ C_{min},\ \frac{\beta_o}{2}\}$.

5 Numerical Experiments

Computationally, we adopt the following strategy. First we solve the local problems 11 to obtain $\{\boldsymbol{\sigma}_h, \mathbf{u}_h\}$ in terms of $\hat{\mathbf{u}}_h$ and to assemble the global equation 12 with Lagrange multipliers only. After solving the global system in $\hat{\mathbf{u}}$, the local problems are revisited to compute $\{\boldsymbol{\sigma}_h, \mathbf{u}_h\}$ element-by-element. A serial implementation was used. In our numerical experiments we observed that the local problems have a marginal computational cost compared to the cost of the global system.

The selected problem is the plane state of stress defined in the domain $\Omega = (0,1)\times(0,1)$ with prescribed displacement on all boundary $\partial\Omega$, $\mathbf{g}_D = 0$. The material properties are Young's *modulus* $E = 1$ and Poisson's ratio $\nu = 0,3$. Choosing appropriately the body force $\mathbf{f}(\mathbf{x}, \mathbf{y})$ so that the exact solution $\mathbf{u} = \{u_x, u_y\}$ of the problem is given by $u_x(x,y) = 10(y - y^2)\sin(\pi x)(1-x)(1-\frac{y}{2})$ and $u_y(x,y) = 0$.

In this study we compare the SHDM-Finite Element Approximation with the Interpolant Approximation and consider the parameters $\delta_1 = \frac{1}{2}$, $\delta_2 = -\frac{1}{2}$ and two cases for stabilization parameter, $\beta_o = 1$ and $\beta_o = 0$. The study of h-convergence is presented for same polynomial order for all fields, $k = l = m = 3$. We use a sequence of mesh 4×4, 8×8, 16×16, and 32×32 with quadrilateral elements.



(a) displacement $\beta_o = 1$ (b) stress $\beta_o = 1$ (c) displacement $\beta_o = 0$ (d) stress $\beta_o = 0$

Figure 1: h-Convergence of Stabilized Hybrid Dual-Mixed Approximation compared with the Interpolant Approximation to polynomial order k=l=m=3 and different value for stabilization parameter β_o.

The numerical evidences given by Figure 1 show optimal rate of convergence to both approximate solutions and its gradients when the stabilization parameter $\beta_o = 1$

(*e.g.* $k = l = m = 3$). However, when $\beta_o = 0$ (local conservation condition) the formulation is stable but the stress approximation converge between optimal and suboptimal rate and the displacement approximation continues with optimal convergence rate.

6 Concluding Remarks

We propose a Stabilized Hybrid Dual-Mixed Finite Element Approximation for the linear elasticity problem. The global problem is assembled in the Lagrange multiplier only and then local problems are used to recover displacement and stress approximations in the interior of the element. Numerical evidences indicate that the approximate fields converge with optimal rate for $\beta_o = 1$ while for $\beta_o = 0$ the stress approximation converge with nearly optimal convergence rates. Many others important features of this formulation will be presented in future works.

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