Proceeding Series of the Brazilian Society of Applied and Computational Mathematics, Vol. 5, N. 1, 2017.

Trabalho apresentado no CNMAC, Gramado - RS, 2016.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

Error analysis for a new mixed finite element method in 3D

Douglas A. Castro[1]
Universidade Federal do Tocantins - Campus Gurupi, TO
Philippe R. B. Devloo [2]
Faculdade de Engenharia Civil Arquitetura e Urbanismo - Unicamp, Campinas, SP
Agnaldo M. Farias[3]
Departamento de Matemática - IFNMG, Salinas, MG
Sônia M. Gomes[4]
Instituto de Matemática, Estatística e Computação Científica, Unicamp, Campinas, SP
Denise de Siqueira[5]
Departamento de Matemática, UTFPR, Curitiba, PR

Abstract. There are different possibilities of choosing balanced pairs of approximation spaces for dual (flux) and primal (pressure) variables; to be used in discrete versions of the mixed finite element method for elliptic problems arising in fluid simulations. Three cases shall be studied for discretized three dimensional formulations, based on tetrahedral, hexahedral, and prismatic meshes. The principle guiding the constructions of the approximation spaces is the property that, the divergence of the dual space and the primal approximation space, should coincide, while keeping the same order of accuracy for the flux variable, and varying the accuracy order of the primal variable. Some cases correspond either to the classic spaces of Raviart-Thomas, Brezzi-Douglas-Marini, Brezzi-Douglas-Fortin-Marini or Nédélec types. A new kind of approximation is proposed by further incrementing the order of some internal flux functions, and matching primal functions at the border fluxes. In this article we develop a unified error analysis for all these space families, and element geometries.

Keywords. $\mathbf{H}$(div) spaces, mixed formulation, approximation space configurations, convergence rates, $3D$ meshes.

1 Introduction

Mixed finite element methods have the ability to provide accurate and locally conservative fluxes, an advantage over standard H^1-finite element discretizations [3]. They are based on simultaneous approximations of the primal (pressure p) and dual (flux $\boldsymbol{\sigma}$) variables, involving two kinds of approximation spaces $(\mathbf{V}_h, U_h)$. For the present analysis, they are supposed to be piecewise defined on affine partitions $\Gamma_h = \{K\}$ of the computational

[1]dacastro@mail.uft.edu.br
[2]phil@fec.unicamp.br
[3]agnaldo.farias@ifnmg.edu.br
[4]soniag@ime.unicamp.br
[5]denisesiqueira@utfpr.edu.br

domain Ω by either tetrahedral, hexahedral or prismatic elements K. For the definitions of the approximation spaces, some general aspects are usually taken into account:

- There are spaces $P_k(K)$ of polynomials restricted to K used in the construction of scalar approximations for the primal variable, where the index k refers to the polynomial degree. For tetrahedra, the polynomials in $P_k(K)$ have total degree k, and for hexahedra, they have maximum degree k in each coordinate. For prismatic elements, $P_k(K)$ is formed by polynomials of total degree k in the triangular faces, and of maximum degree k in the complementary direction. Vectorial polynomial spaces $\mathbf{P}_k(K)$ mean that the components of the vectorial shape functions are obtained from polynomials in $P_k(K)$.

- The approximation subspaces $U_h \subset L^2(\Omega)$ for the primal variable are piecewise formed as $u|_K = u_K$, for $K \in \Gamma_h$, without any continuity constraint.

- The approximation subspaces $\mathbf{V}_h \subset \mathbf{H}(\text{div}, \Omega)$ are formed by functions $\mathbf{q}$ piecewise defined over the elements of Γ_h by local functions $\mathbf{q}_K = \mathbf{q}|_K$, which are spanned by hierarchical vectorial bases in $\mathbf{H}(\text{div}, K)$. The shape functions $\mathbf{\Phi}$ in those bases can be classified as of interior type, with vanishing normal components over all element faces. Otherwise, $\mathbf{\Phi}$ is classified as of face type, with normal components vanishing over all other faces not associated to it.

- In all the cases, the choices of the approximation spaces are guided by the property

$$\nabla \cdot \mathbf{V}_h = U_h, \tag{1}$$

  in order to obtain stable results with optimal L^2-error convergence orders, which are dictated by the degree of the complete set of polynomials used to form the corresponding approximation spaces.

Three cases shall be studied. Some correspond to classic Raviart-Thomas [6, 8], Brezzi-Douglas-Marini [1], Brezzi-Douglas-Fortin-Marini [2], and Nédélec [7] space types. A new kind of approximations is proposed by further incrementing the order of some internal flux functions, and matching primal functions at the border fluxes. The main purpose is to develop a unified error analysis for the mixed method based on all these families of space configurations, and element geometries.

In Section 2, the mixed finite element method is set for a model problem, and the three kinds of space configurations are described. Section 3 is dedicated to summarize the classic techniques that can be used in a unified way for stability and error analyses of approximate solutions $(\boldsymbol{\sigma}_h, u_h) \in \mathbf{V}_h \times U_h$ of the mixed formulation based on such spaces. For this end, it is necessary to define projections commuting the de Rham diagram, which is the subject developed in Section 4. Summarizing conclusions are given in Section 5.

2 Mixed finite element method for a model problem

As studied in [3], $\mathbf{H}(\text{div})$-conforming discretized versions of the mixed formulation search for approximate solutions $\boldsymbol{\sigma}_h$ and u_h in finite dimensional subspaces $\mathbf{V}_h \subset \mathbf{H}(\text{div}, \Omega)$

and $U_h \subset L^2(\Omega)$, such that for all $\mathbf{q} \in \mathbf{V}_h$, and $\varphi \in U_h$

$$\int_\Omega \boldsymbol{\sigma}_h \cdot \mathbf{q} \, d\Omega - \int_\Omega u_h \nabla \cdot \mathbf{q} \, d\Omega = 0, \quad (2)$$

$$\int_\Omega \nabla \cdot \boldsymbol{\sigma}_h \varphi \, d\Omega = \int_\Omega f \, \varphi \, d\Omega. \quad (3)$$

Three kinds of space configurations for $(\mathbf{V}_h, U_h)$ shall be considered in this article.

Approximations of type $\mathbf{P}_k$ P_{k-1}. For this configuration, the dual approximation space of type $\mathbf{P}_k$ is based on a complete vector valued polynomials of degree k, and the primal approximation space of type P_{k-1} is based on the complete scalar valued polynomials of degree $k-1$. However, it is well known that this kind of space configuration satisfying property (1) can only be valid for triangular and tetrahedral meshes, corresponding to the classic BDM_k elements, for which L^2-error convergence of orders $k+1$ and k, for dual and primal variables can be obtained.

Approximations of type $\mathbf{P}_k^*$ P_k. For all the geometries, another type of space configuration can be considered. Guided by the verification of property (1), the dual approximations are said to be of $\mathbf{P}_k^*$ type if are locally spanned by the face functions of $\mathbf{P}_k(K)$ type, and by the internal shape functions of $\mathbf{P}_{k+1}(K)$, defined by a vectorial polynomials of degree $k+1$ whose divergence are included in the primal approximation space of type $P_k(K)$. Since the incomplete dual approximation space of type $\mathbf{P}_k^*$ only involves the complete vector valued polynomials of degree k, in simulations using $\mathbf{P}_k^*$ P_k configurations, the expected L^2-error convergence rates are of order $k+1$ for, both dual and primal variables. This is the type of RT_k space configuration for quadrilateral geometries, and their generalization to hexahedral partitions in [6], and of $BDFM_{k+1}$ elements for triangular or tetrahedral elements. For prismatic elements, $\mathbf{P}_k^*(K)$ contains Nédélec's elements $N_k(K)$ [7].

Approximations of type $\mathbf{P}_k^{**}$ P_{k+1}. The construction of flux approximation spaces of type $\mathbf{P}_k^{**}$ consists in adding to the complete vector valued spaces of type $\mathbf{P}_k$ those interior shape functions of $\mathbf{P}_{k+1}^*(K)$ defined by vectorial polynomials of degree $k+2$ whose divergence are included in the primal approximation space of type $P_{k+1}(K)$. Therefore, in $\mathbf{P}_k^{**}(K)$, the face shape functions are still obtained by polynomials of degree $\leq k$, but some of the internal shape functions may be obtained from polynomials of degree up to $k+2$. As in the previous case, the verification of property (1) is the basic principle guiding the definition of the pair of approximation spaces. For this setting, $\mathbf{P}_k^{**}(K)$ contains only the complete vector valued approximations $\mathbf{P}_k(K)$, and the L^2-error convergence rate of order $k+1$ is expected for the dual variable. However, for primal variable, a higher order $k+2$ may be reached. To our knowledge, space configurations $\mathbf{P}_k^{**}$ P_{k+1}, which are valid for all element geometries, are new in the literature.

DOI: 10.5540/03.2017.005.01.0331

3 Error estimates

Similar to the unified error analysis described in [3] for a variety of approximation settings, let us denote by $\mathbf{M}(K)$ any of the local approximation spaces either of type $\mathbf{P}_k(K)$, $\mathbf{P}^*_k(K)$ or $\mathbf{P}^{**}_k(K)$, restricted to elements $K \in \Gamma_h$. We have, $\mathbf{P}_k(K) \subsetneqq \mathbf{M}(K)$ and $\mathbf{P}_{k+1}(K) \nsubseteq \mathbf{M}(K)$. Let also consider scalar spaces $D(K) = \nabla \cdot \mathbf{M}(K)$. For thetrahedra, $D(K) = P_{k-1}(K)$ for the BDM_k spaces, where $\mathbf{M}(K) = \mathbf{P}_k(K)$. For all geometries, $D(K) = P_k(K)$ for spaces $\mathbf{M}(K) = \mathbf{P}^*_k(K)$, and $D(K) = P_{k+1}(K)$ when $\mathbf{M}(K) = \mathbf{P}^{**}_k(K)$. Consider pairs of spaces $\mathbf{V}_h \times U_h$ defined by local approximations

$$\mathbf{V}_h = \{\mathbf{q} \in \mathbf{H}(\mathrm{div}, \Omega);\ \mathbf{q}|_K \in \mathbf{M}(K),\ K \in \Gamma_h\},\ \ U_h = \left\{u \in L^2(\Omega);\ u|_K \in D(K),\ K \in \Gamma_h\right\}.$$

By construction, the crucial property $\nabla \cdot \mathbf{V}_h = U_h$ holds. Therefore, classic techniques can be used for stability and error analyses of approximate solutions $(\boldsymbol{\sigma}_h, u_h) \in \mathbf{V}_h \times U_h$ of the mixed formulation based on such spaces. For this end, it is necessary to define projections

$$\Lambda_h \times \Pi_h : \mathbf{H}^1(\Omega) \times L^2(\Omega) \to \mathbf{V}_h \times U_h$$

commuting the de Rham diagram

$$\begin{array}{ccc} \mathbf{H}^1(\Omega) & \xrightarrow{\nabla\cdot} & L^2(\Omega) \\ \downarrow \Lambda_h & & \downarrow \Pi_h \\ \mathbf{V}_h & \xrightarrow{\nabla\cdot} & U_h. \end{array}$$

The functional space $\mathbf{H}^1(\Omega) \subset \mathbf{H}(\mathrm{div}, \Omega)$ denotes the vector space analogue of $H^1(\Omega)$, which is used in order to guarantee L^2-integrable normal traces $\mathbf{q} \cdot \boldsymbol{\eta}_K|_{\partial K}$ over element boundaries.

For $z \in L^2(\Omega)$, the projection $\Pi_h z$ on the scalar approximation space U_h is usually taken as the L^2-projection such that

$$\int_\Omega (z - \Pi_h z)\varphi d\Omega = 0,\ \forall \varphi \in U_h.$$

For smooth vectorial functions $\mathbf{q} \in \mathbf{H}^1(\Omega) \subset \mathbf{H}(\mathrm{div}, \Omega)$, the projection $\Lambda_h \mathbf{q}$ can be defined in terms of local projections by

$$\Lambda_h \mathbf{q}|_K = \lambda_K \mathbf{q},\ \forall K \in \Gamma_h,$$

where $\lambda_K : \mathbf{H}^1(K) \to \mathbf{M}(K)$ should verify the local de Rham property

$$\nabla \cdot \lambda_K \mathbf{q} = \pi_K \nabla \cdot \mathbf{q}, \tag{4}$$

and π_K denotes the L^2-projetion on $D(K)$. Based on such properties, Proposition 1.2, p. 139 in [3] establishes convergence rates of the form

$$||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_0 = O(h^{k+1}), \tag{5}$$

$$||u - u_h||_0 = O(h^s), \tag{6}$$

with $s = k$ for BDM_k spaces of type $\mathbf{P}_k$ P_{k-1} for tetrahedral partitions, $s = k+1$ when spaces of type $\mathbf{P}_k^*$ P_k are used, for tetrahedral $BDFM_{k+1}$ elements, and hexahedral Raviart-Thomas elements RT_k. Here $||.||_0$ denotes either the usual L^2-norms of vectorial or scalar spaces. In addition, because the meshes are supposed to be affine, and from the fact that $\nabla\cdot\boldsymbol{\sigma}_h$ is the $L^2(\Omega)$-projection of $\nabla\cdot\boldsymbol{\sigma}$ on U_h, the divergence error $||\nabla\cdot\boldsymbol{\sigma} - \nabla\cdot\boldsymbol{\sigma}_h||_{L^2(\Omega)}$ has the same accuracy rate as the error in u.

In the next section, local projections $\lambda_K\mathbf{q}$ commuting the local de Rham diagram shall be defined. Based on their properties, similar convergence rates hold for the new spaces of type $\mathbf{P}_k^*$ P_k based on prisms, as in $BDFM_{k+1}$ for tetrahedra, and RT_k for hexahedral meshes. Furthermore, the main result of this paper concerns error estimates for the new $\mathbf{P}_k^{**}$ P_{k+1} space configurations. With respect to the accuracy of order $k+1$ for the flux, we conclude that enhanced accuracy order $s = k+2$ for the primal variable can be verified for all element geometries.

4 Local projections commuting the de Rham diagram

This part is dedicated to the study of the required local projections $\lambda_K\mathbf{q}$ verifying the commuting de Rham diagram property (4), for all spaces $\mathbf{M}(K)$ either of type $\mathbf{P}_k(K)$, $\mathbf{P}_k^*(K)$ or $\mathbf{P}_k^{**}(K)$, described in Section 2, which are crucial for the error analysis.

For all cases, let the space

$$P(\partial K) = \left\{\phi \in L^2(\partial K); \phi|_F \in P_k(F) \text{ on the faces } F \text{ of } K\right\}$$

represents the normal traces of functions in $\mathbf{M}(K)$. Consider direct decompositions $\mathbf{M}(K) = \mathbf{M}_\partial(K) \oplus \mathring{\mathbf{M}}(K)$, where $\mathring{\mathbf{M}}(K) = \{\boldsymbol{\sigma} \in \mathbf{M}(K);\ \boldsymbol{\sigma}\cdot\boldsymbol{\eta}_K|_{\partial K} = 0\}$ is the space of internal functions in $\mathbf{M}(K)$, and $\mathbf{M}_\partial(K)$ being its complement.

Following the suggestions in [5], let us consider local projections factorized as $\lambda_K\mathbf{q} = \mathbf{q}_\partial + \mathring{\mathbf{q}}$, with boundary terms $\mathbf{q}_\partial \in \mathbf{M}_\partial(K)$, and internal terms $\mathring{\mathbf{q}} \in \mathring{\mathbf{M}}(K)$. The following observations are in order for the factorizations of different settings of $\mathbf{M}(K)$.

1. The boundary contributions $\mathbf{q}_\partial$ can be computed in terms of the face shape functions in $\mathbf{P}_k(K)$, for all the cases.

2. The specifications for the internal contributions $\mathring{\mathbf{q}}$ depends on the specific space configuration adopted.

For the tetrahedral elements there are three settings:

1. $\mathbf{M}(K) = \mathbf{P}_k(K)$, $D(K) = P_{k-1}(K)$: for this case, denote $\lambda_K = \rho_{k,K}$ for the classic BDM_k projection on $\mathbf{P}_k(K)$. Then $\mathring{\mathbf{q}} = \lambda_K(\mathbf{q} - \mathbf{q}_\partial) \in \mathbf{P}_k(K)$, with $\mathring{\mathbf{q}}\cdot\boldsymbol{\eta}_K|_{\partial K} = 0$, can be computed in terms of the internal shape functions in $\mathbf{P}_k(K)$.

2. $\mathbf{M}(K) = \mathbf{P}_k^*(K)$ and $D(K) = P_k(K)$: in this case, let $\lambda_K = \rho_{k,K}^*$ be the classic $BDFM_{k+1}$ projection on $\mathbf{P}_k^*(K)$. Then, $\mathring{\mathbf{q}} = \mathring{\mathbf{q}}^* = \rho_{k,K}^*(\mathbf{q} - \mathbf{q}_\partial) \in \mathbf{P}_k^*(K)$ with $\mathring{\mathbf{q}}^*\cdot\boldsymbol{\eta}_K|_{\partial K} = 0$. Since the internal spaces in $\mathbf{P}_k^*(K)$ and in $\mathbf{P}_{k+1}(K)$ coincide for tetrahedral elements, then this internal contribution can also be represented as $\mathring{\mathbf{q}}^* =$

$\rho_{k+1,K}(\mathbf{q}-\mathbf{q}_\partial)$, in terms of the BDM_{k+1} projection on $\mathbf{P}_{k+1}(K)$, which can be expressed in terms of the internal shape functions of $\mathbf{P}_{k+1}(K)$.

3. $\mathbf{M}(K) = \mathbf{P}_k^{**}(K)$ and $D(K) = P_{k+1}(K)$: since for tetrahedral elements the three internal spaces in $\mathbf{P}_k^{**}(K)$, $\mathbf{P}_{k+1}^{*}(K)$ and $\mathbf{P}_{k+2}(K)$ coincide, for this particular setting we propose to define $\lambda_K = \rho_{k,K}^{**}$ as the projection such that $\mathring{\mathbf{q}}^{**} = \rho_{k+1,K}^{*}(\mathbf{q}-\mathbf{q}_\partial) = \rho_{k+2,K}(\mathbf{q}-\mathbf{q}_\partial)$, with $\mathring{\mathbf{q}}^{**}\cdot\boldsymbol{\eta}_K|_{\partial K} = 0$. Thus, $\mathring{\mathbf{q}}^{**}$ can be computed by either the $BDFM_{k+1}$ or by the BDM_{k+2} projections, meaning that it can be expressed by the internal shape functions of $\mathbf{P}_{k+2}(K)$.

For hexahedral and prismatic elements there are two different settings:

1. $\mathbf{M}(K) = \mathbf{P}_k^{*}(K)$ and $D(K) = P_k(K)$. For hexahedral elements: the projection $\lambda_K = \rho_{k,K}^{*}$ is taken by the classic RT_k projection, as proposed in [6], for which $\mathring{\mathbf{q}}^{*} = \rho_{k,K}^{*}(\mathbf{q}-\mathbf{q}_\partial) \in \mathbf{P}_k^{*}(K)$, with $\mathring{\mathbf{q}}^{*}\cdot\boldsymbol{\eta}_K|_{\partial K} = 0$. For prismatic elements, this framework shares with the previous Nédélec's setting the scalar spaces $D(K)$ and $P(\partial K)$, and the boundary space $\mathbf{M}_\partial(K)$ as a consequence. Since $\mathring{\mathbf{M}}(K)$ strictly contains the internal Nédélec's functions, a projection $\lambda_K = \rho_{k,K}^{*}$ on $\mathbf{P}_k^{*}$ commuting the de Rham diagram can be derived from the Nédélec's projection, similarly as in Appendix B of [4], where the case of RT spaces of variable degree on a simplex is considered.

2. $\mathbf{M}(K) = \mathbf{P}_k^{**}(K)$ and $D(K) = P_{k+1}(K)$, for hexahedral and prismatic elements: since the internal spaces in $\mathbf{P}_k^{**}(K)$ coincide with the corresponding internal spaces in $\mathbf{P}_{k+1}^{*}(K)$, assuming projections $\rho_{k,K}^{*}$ being stated, $\lambda_K = \rho_{k,K}^{**}$ is defined by taking $\mathring{\mathbf{q}}^{**} = \rho_{k+1,K}^{*}(\mathbf{q}-\mathbf{q}_\partial)$, with $\mathring{\mathbf{q}}^{**}\cdot\boldsymbol{\eta}_K|_{\partial K} = 0$, which can be expanded by the internal shape functions of $\mathbf{P}_{k+1}^{*}(K)$. Recalling the definition $\rho_{k,K}^{**}\mathbf{q} = \mathbf{q}_\partial + \mathring{\mathbf{q}}^{**}$, we obtain $\nabla\cdot\rho_{k,K}^{**}\mathbf{q} = \nabla\cdot\mathbf{q}_\partial + \nabla\cdot\rho_{k+1,K}^{*}(\mathbf{q}-\mathbf{q}_\partial)$. Since $\mathbf{q}_\partial \in \mathbf{P}_k$, we get $\nabla\cdot\mathbf{q}_\partial = \pi_{k+1,K}\nabla\cdot\mathbf{q}_\partial$, and using the commuting de Rham property $\rho_{k+1,K}^{*}\ \nabla\cdot\rho_{k+1}^{*}(\mathbf{q}-\mathbf{q}_\partial) = \pi_{k+1,K}\nabla\cdot(\mathbf{q}-\mathbf{q}_\partial)$, which is valid for all projections $\rho_{k+1,K}^{*}$ associated to the settings $\mathbf{M}_{k+1}(K) = \mathbf{P}_{k+1}^{*}(K)$ and $D_{k+1}(K) = P_{k+1}(K)$, then the desired commuting property holds for $\rho_{k,K}^{**}\mathbf{q}$ as well.

5 Conclusions

Different choices of balanced finite element approximation spaces for dual and primal variables, based on tetrahedral, hexahedral and prismatic meshes, are considered for discrete versions of the mixed finite element method for three dimensional elliptic problems. The principle guiding the constructions of the approximations is the property that, the divergence of the dual space and the primal approximation space, should coincide, while keeping the same order of accuracy for the flux variable and varying the accuracy order of the primal variable. In all these three settings, the degrees of freedom associated with internal flux functions can be condensed. Therefore, for each element geometry, and fixed degree k used in the border flux approximations, the resulting global condensed matrices have identical sizes. Convergence studies are presented to show optimal rates in

L^2-norms for primal and dual variables, which are determined by the degree of the complete polynomial spaces, included in the corresponding approximations spaces. In fact, the convergence rates for the dual variable do not change by increasing the degrees of internal flux functions, being the same computational cost to obtain higher optimal convergence rates.

Acknowledgements

The authors P.R.B. Devloo, and S.M. Gomes thankfully acknowledges financial support from ANP-Brazilian National Agency of Petroleum, Natural Gas and Biofuels, and from CNPq, the Brazilian Research Council. D.A. Castro is grateful for the hospitality received during his pos-doc visit at LabMeC-FEC-Unicamp, when this paper was prepared, and for financial support received from FAPESP-Research Foundation of the State of São Paulo, Brazil. A.M. Farias thankfully acknowledge financial support from ANP and from CAPES Foundation, within the Ministry of Education in Brazil.

References

[1] F. Brezzi, J. Douglas, L.D. Marini, Two Families of Mixed Finite Elements for Second Order Elliptic Problems. *Numer. Math.* 47 (1985) 217–235.

[2] F. Brezzi, J. Douglas, M. Fortin, L.D. Marini, Efficient rectangular mixed finite elements in two and three space variable. *RAIRO Modél. Math. Anal. Numér.* 2 (1987) 581–604.

[3] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods. *Springer Series in Computational Mathematics*, 15, Springer-Verlag, NewYork, 1991.

[4] B. Cockburn, J. Gopalakrishnan, Error analysis of variable degree mixed methods for elliptic problems via hybridization. *Math. Comput.* 74 (252) (2005) 1653–1677.

[5] L. Demkowicz, P. Monk, L. Vardapetvan, W. Rachowicz, De Rham diagram for *hp* finite element spaces. *Comput. Math. Appl.* 39 (2000) 29–38.

[6] J.C. Nédélec, Mixed finite elements in $\mathbb{R}^3$. *Numer. Math.* 35 (1980) 315–341.

[7] J.C. Nédélec, A New family of mixed finite elements in $\mathbb{R}^3$, *Numer. Math.* 50 (1986) 57–81.

[8] P.A. Raviart, J.M. Thomas, A mixed finite element method for 2nd order elliptic problems, *Lect. Not. Math.* 606 (1977) 292–315.

[9] D. Siqueira, P.R.B. Devloo, S.M. Gomes, A new procedure for the construction of hierarchical high order Hdiv and Hcurl finite element spaces, *J. Comput. Appl. Math.* 240 (2013) 204–214.

