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A Stabilized Hybrid Discontinuous Galerkin method for linear elasticity problem

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Abstract. In this work, a primal hybrid finite element method for nearly incompressible linear elasticity problem on triangular meshes is shown. This method consists of coupling local discontinuous Galerkin problems to the primal variable with a global problem for the Lagrange multiplier, which is identified as the trace of the displacement field. Also, a local post-processing technique is used to recover stress approximations with improved rates of convergence in H(div) norm. Numerical studies show that the method is locking free even using equal or different orders for displacement and stress fields and optimal convergence rates are obtained.

Keywords. Linear elasticity, Discontinuous Galerkin method, Stabilization Hybrid method, Locking free

1 Introduction

Discontinuous Galerkin (DG) methods are naturally a suitable alternative for solving linear elasticity problems. Robustness, local conservation and flexibility for implementing h and p-adaptivity strategies are well known advantages of DG methods stemming from the use of finite element spaces consisting of discontinuous piecewise polynomials. A natural connection between DG formulations and hybrid methods have been exploited successfully in many problems [1–4] and they are still being developed. These hybrid formulations have improved stability, robustness and flexibility of the DG methods with reduced complexity and computational cost. Hybrid finite element methods are characterized by the introduction of new unknowns variables, the Lagrange multipliers, defined an the edges of the elements to weakly impose continuity on the element interfaces. Their reduced complexity and computational cost is obtained by eliminating all degrees of freedom of the primal variables at the element level resulting a global system only with degrees of freedom of the multipliers.

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In this paper, we use a primal Stabilized Hybrid DG (SHDG) method, based in [4, 5], for the linear elasticity problem considering triangular elements. The multiplier, identified with the trace of the displacement field, is interpolated discontinuously. The method is stable for any order of interpolation of the displacement field and the multiplier. Stress approximations are recovered by a local post-processing for both displacement and stress field using the multiplier approximation obtained by the SHDG method. The residual form of the equilibrium equation at the element level is added in this local post-processing leading to optimal rates of convergence of the stress approximations in H(div) norm for compressible and nearly incompressible elasticity problems with uniform meshes.

2 The Model Problem

Let Ω in $\mathbb{R}^2$ an open bounded domain with piecewise Lipschitz boundary $\Gamma_D = \partial\Omega$ of an elastic body subjected to external force $\mathbf{f} \in [L^2(\Omega)]^2$ where Γ_D is the Dirichlet boundary. The kinematical model of linear elasticity problem in two dimensions consists in finding a displacement vector field $\mathbf{u}$ satisfying

$$\begin{aligned} -\operatorname{div}\boldsymbol{\sigma}(\mathbf{u}) &= \mathbf{f} && \text{in } \Omega, \\ \boldsymbol{\sigma}(\mathbf{u}) &= \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}) && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{g} && \text{on } \Gamma_D, \end{aligned} \tag{1}$$

where $\boldsymbol{\sigma}(\mathbf{u})$ is the symmetric Cauchy stress tensor, $\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\operatorname{grad}\mathbf{u} + \operatorname{grad}\mathbf{u}^T)$ is the linear strain tensor, $\mathbf{g}$ is a given boundary displacement.

For linear, homogeneous and isotropic material $\boldsymbol{\sigma}(\mathbf{u})$ is given by $\boldsymbol{\sigma}(\mathbf{u}) = \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}) = 2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda(\operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{u}))\mathbb{I}$, where $\operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{u}) = \operatorname{div}\mathbf{u}$, $\mathbb{I}$ is the identity tensor and λ and μ are called the Lamé parameters which are given in terms of elasticity modulus E and Poisson ratio ν by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)}.$$

Nearly incompressible materials are modeled by Poisson ratio when $\nu \to 1/2$. It is well known that in this limit classical Galerkin FEM approximations lead to volume locking.

3 The SHDG Formulation

In this section we present the Stabilized Hybrid Discontinuous Galerkin (SHDG) formulation for the linear elasticity problem based in [4]. In this primal hybrid formulation the multiplier $\boldsymbol{\lambda}$ is identified as the trace of the displacement field. That is, $\mathbf{u}$: $\boldsymbol{\lambda} = \mathbf{u}|_e$ on each edge $e \in \mathcal{E}_h$.

To construct the primal hybrid formulation we introduce the following broken function space for the displacement field

$$\mathbf{V}_h^k = \{\mathbf{v}_h \in \mathbf{L}^2(\Omega) \;:\; \mathbf{v}_h|_K \in [S_k(K)]^2 \quad \forall K \in \mathcal{T}_h\}, \tag{2}$$

where $S_k(K) = P_k(K)$ (the space of polynomial functions of degree at most k in both variables) and for the multiplier

$$\mathbf{M}_h^l = \{\boldsymbol{\lambda} \in \mathbf{L}^2(\mathcal{E}_h) : \boldsymbol{\lambda}|_e = [p_l(e)]^2,\ \forall e \in \mathcal{E}_h^0\}, \tag{3}$$

of polynomial $p_l(e)$, of degree at most l, on each edge e.

In these finite dimension spaces, the SHDG method is formulated as:
Find the pair $[\mathbf{u}_h, \boldsymbol{\lambda}_h] \in \mathbf{V}_h^k \times \mathbf{M}_h^l$ such that, for all $[\mathbf{v}_h, \boldsymbol{\mu}_h] \in \mathbf{V}_h^k \times \mathbf{M}_h^l$

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_h} \int_K \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h) : \boldsymbol{\varepsilon}(\mathbf{v}_h)dx - \sum_{K\in\mathcal{T}_h} \int_{\partial K} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h)\mathbf{n}_K \cdot (\mathbf{v}_h - \boldsymbol{\mu}_h)ds + \\
&\quad - \sum_{K\in\mathcal{T}_h} \int_{\partial K} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{v}_h)\mathbf{n}_K \cdot (\mathbf{u}_h - \boldsymbol{\lambda}_h)ds + 2\mu \sum_{K\in\mathcal{T}_h} \beta \int_{\partial K} (\mathbf{u}_h - \boldsymbol{\lambda}_h) \cdot (\mathbf{v}_h - \boldsymbol{\mu}_h)ds \\
&\quad = \sum_{K\in\mathcal{T}_h} \int_K \mathbf{f} \cdot \mathbf{v}_h dx,
\end{aligned} \tag{4}$$

where the stabilization parameter β depends on the mesh parameter h. Here, we have considered the following definition for this stabilization parameter $\beta = \frac{\beta_0}{h}, \forall e \in \mathcal{E}_h$ with $\beta_0 > 0$. The influence of β_0 on the stability and accuracy of this primal hybrid formulation is analyzed in [4] for compressible elasticity problems.

4 Computational Implementation

Considering that $\mathbf{v}_h$, belonging to the broken function space $\mathbf{V}_h^k$, is defined independently on each element $K \in \mathcal{T}_h$, we observe that equation (4) can be split into a set of local problems defined on each element K coupled to the global problem defined on $\mathcal{E}_h$, as follow:

Local problems: Find $\mathbf{u}_h|_K \in \mathbf{V}_h^k(K) = \mathbf{V}_h^k|_K$, such that, for all $\mathbf{v}_h|_K \in \mathbf{V}_h^k(K)$,

$$\begin{aligned}
&\int_K \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h) : \boldsymbol{\varepsilon}(\mathbf{v}_h)dx - \int_{\partial K} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h)\mathbf{n}_K \cdot \mathbf{v}_h ds - \int_{\partial K} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{v}_h)\mathbf{n}_K \cdot (\mathbf{u}_h - \boldsymbol{\lambda}_h)ds + \\
&\quad + 2\mu \int_{\partial K} \beta(\mathbf{u}_h - \boldsymbol{\lambda}_h) \cdot \mathbf{v}_h ds = \int_K \mathbf{f} \cdot \mathbf{v}_h dx,
\end{aligned} \tag{5}$$

Global Problem: Find $\boldsymbol{\lambda}_h \in \mathbf{M}_h^l$, such that, for all $\boldsymbol{\mu}_h \in \mathbf{M}_h^l$,

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h)\mathbf{n}_K \cdot \boldsymbol{\mu}_h ds - \sum_{K\in\mathcal{T}_h} 2\mu \int_{\partial K} \beta(\mathbf{u}_h - \boldsymbol{\lambda}_h) \cdot \boldsymbol{\mu}_h ds = 0,\ \forall \boldsymbol{\mu}_h \in \mathbf{M}_h^l. \tag{6}$$

Given that the multiplier of the proposed hybrid formulation is identified with the trace of the primal variable $\mathbf{u}$ on the element edges, for appropriate choices of β, we can always eliminate the degrees-of-freedom of the primal variable $\mathbf{u}_h$ at the element level in favor of the degrees-of-freedom of the multiplier leading to a global system in the multiplier only. Note that we can adopt any (l) order of continuous or discontinuous interpolation functions to Lagrange multiplier $\boldsymbol{\lambda}_h$ independently of the (k) order adopted for the primal variable $\mathbf{u}_h$. Here, we consider only discontinuous interpolations for the Lagrange multiplier.

4.1 Stress and displacement local post-processing

In most engineering applications, stresses are the variables of main interest. Classically, in displacement finite element formulation, stresses are computed indirectly using the displacement approximation and the constitutive equation only. With this classical approach, the stress approximation is given by

$$\boldsymbol{\sigma}_h = \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h) \tag{7}$$

which converges at best with the following rates in $\mathbf{L}^2$ and $\mathbf{H}(\mathrm{div})$ norms:

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{L}^2} = \|\mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}) - \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_h)\|_{\mathbf{L}^2} = Ch^k, \tag{8}$$

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{H}(\mathrm{div})} = Ch^{k-1}. \tag{9}$$

As an improved alternative to compute stress approximations, we propose a local post-processing consisting in solving at each element $K \in \mathcal{T}_h$ the local problem:

$$\begin{array}{rll} -\operatorname{div}\boldsymbol{\sigma}(\mathbf{u}) & = \mathbf{f} & \text{in } K,\\ \mathbb{A}\boldsymbol{\sigma}(\mathbf{u}) & = \boldsymbol{\varepsilon}(\mathbf{u}) & \text{in } K,\\ \mathbf{u} & = \boldsymbol{\lambda}_h & \text{on } \partial K, \end{array} \tag{10}$$

in stress and displacement fields, in which $\mathbb{A} = \mathbb{D}^{-1}$ and $\boldsymbol{\lambda}_h$ is given by the solution of the global problem. Stress and displacement approximations $[\boldsymbol{\sigma}_{pp}, \mathbf{u}_{pp}]$ for $[\boldsymbol{\sigma}, \mathbf{u}]$, solution of (10), are obtained in the finite dimension spaces

$$\mathbb{W}_h^k(K) = \{\tau_{i,j} \in S_k(K), \quad \tau_{i,j} = \tau_{j,i}, \quad i,j = 1,2\} \tag{11}$$

and

$$\mathbf{V}_h^k(K) = \{\mathbf{v}_i \in S_k(K), \quad i = 1,2\}, \tag{12}$$

respectively, considering the following residual form on each element $K \in \mathcal{T}_h$.

Given $\boldsymbol{\lambda}_h$, find $[\boldsymbol{\sigma}_{pp}|_K, \mathbf{u}_{pp}|_K] \in \mathbb{W}_h^k(K) \times \mathbf{V}_h^k(K)$, such that

$$a_{pp}([\boldsymbol{\sigma}_{pp}, \mathbf{u}_{pp}], [\boldsymbol{\tau}_h, \mathbf{v}_h]) = f_{pp}([\boldsymbol{\tau}_h, \mathbf{v}_h]) \quad \forall\ [\boldsymbol{\tau}_h|_K, \mathbf{v}_h|_K] \in \mathbb{W}_h^k(K) \times \mathbf{V}_h^k(K), \tag{13}$$

with

$$\begin{aligned} a_{pp}([\boldsymbol{\sigma}_{pp}, \mathbf{u}_{pp}], [\boldsymbol{\tau}_h, \mathbf{v}_h]) = & \int_K \mathbb{A}\boldsymbol{\sigma}_{pp} : \boldsymbol{\tau}_h dx + \int_K \mathbf{u}_{pp} \cdot \operatorname{div}\boldsymbol{\tau}_h dx + \int_K \operatorname{div}\boldsymbol{\sigma}_{pp} \cdot \mathbf{v}_h dx + \\ & + \frac{\delta}{2\mu}\int_K \operatorname{div}\boldsymbol{\sigma}_{pp} \cdot \operatorname{div}\boldsymbol{\tau}_h dx + 2\mu\int_{\partial K} \beta\ \mathbf{u}_{pp} \cdot \mathbf{v}_h ds, \end{aligned} \tag{14}$$

$$\begin{aligned} f_{pp}([\boldsymbol{\tau}_h, \mathbf{v}_h]) = & \int_{\partial K} \boldsymbol{\lambda}_h \cdot \boldsymbol{\tau}_h \mathbf{n}_K ds - \frac{\delta}{2\mu}\int_K \mathbf{f} \cdot \operatorname{div}\boldsymbol{\tau}_h dx - \int_K \mathbf{f} \cdot \mathbf{v}_h dx + \\ & + 2\mu\int_{\partial K} \beta\ \boldsymbol{\lambda}_h \cdot \mathbf{v}_h ds. \end{aligned} \tag{15}$$

For appropriate choices of the stabilization parameters δ, we have observed the following convergence rate for the post-processed stress:

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_{pp}\|_{\mathbf{H}(\mathrm{div})} = Ch^k \tag{16}$$

which is one order higher than that observed for $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{H}(\mathrm{div})}$.

5 Numerical Results

In this section the behavior of the proposed formulation is tested to plane-strain problem, defined on square domain $\Omega = (0,1) \times (0,1)$ with homogeneous boundary conditions, considering the elasticity modulus $E = 1$ and forcing term:

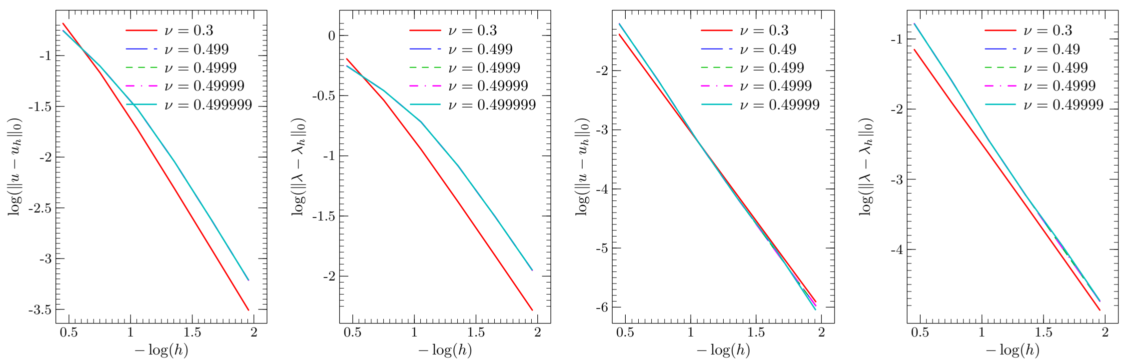
$$f_1(x, y) = (2\nu(2\mu + \lambda) - (\mu + \lambda)) \sin(\pi x) \cos(\pi y), \tag{17}$$

$$f_2(x, y) = (2\nu(2\mu + \lambda) - (3\mu + \lambda)) \sin(\pi y) \cos(\pi x). \tag{18}$$

The approximate solutions have been obtained using triangular elements $P_k - p_l$ where k and l denote the degree of Lagrangian polynomial space for the displacement field and the Lagrange multiplier, respectively. The post-processed stress approximations are recovered with Lagrangian polynomials of degree $k+1$. Next, we present the incompressible elasticity problem, using uniform meshes, comparing the stress approximations σ_{pp} obtained from the local post-processing (13), with σ_h, obtained by the constitutive equation (1).

5.1 h-convergence of displacement field

Figure 1 presents h-convergence studies for $\mathbf{u}_h$ and λ_h of SHDG approximations with equal order ($l = k$) in L^2-norm. In these experiments, we consider the parameters $\beta = 20$ and $\mu = 0.3, 0.49, 0.499, 0.4999, 0.49999$ (nearly incompressible) with triangular elements, $P_1 - p_1$ in Figure 1(a)-(b) and $P_2 - p_2$ in Figure 1(c)-(d). Optimal rates of convergence are observed in $L^2(\Omega)$-norm, with $\mathcal{O}(h^2)$ to $P_1 - p_1$, and $\mathcal{O}(h^3)$ to $P_2 - p_2$ with little loss of precision for nearly incompressible case.

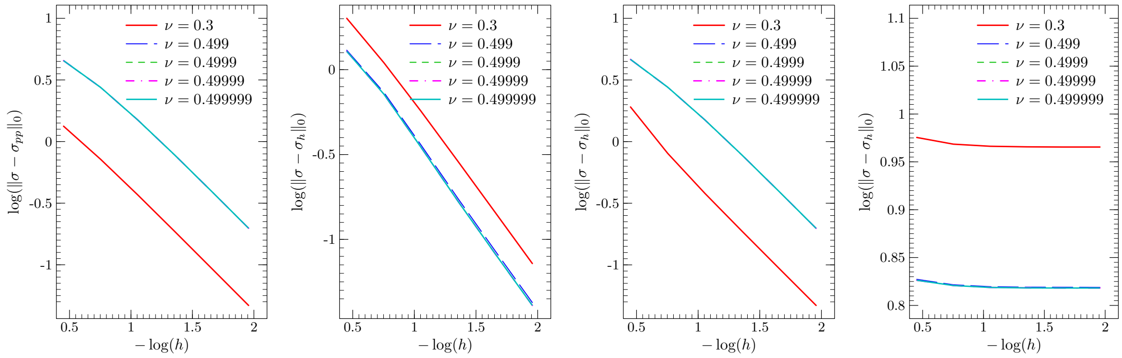


(a) $P_1 - p_1$ (b) $P_1 - p_1$ (c) $P_2 - p_2$ (d) $P_2 - p_2$

Figure 1: h-convergence study for $\mathbf{u}_h$ and λ_h of primal SHDG approximations with $\mu = 0.3, 0.49, 0.499, 0.4999, 0.49999$.

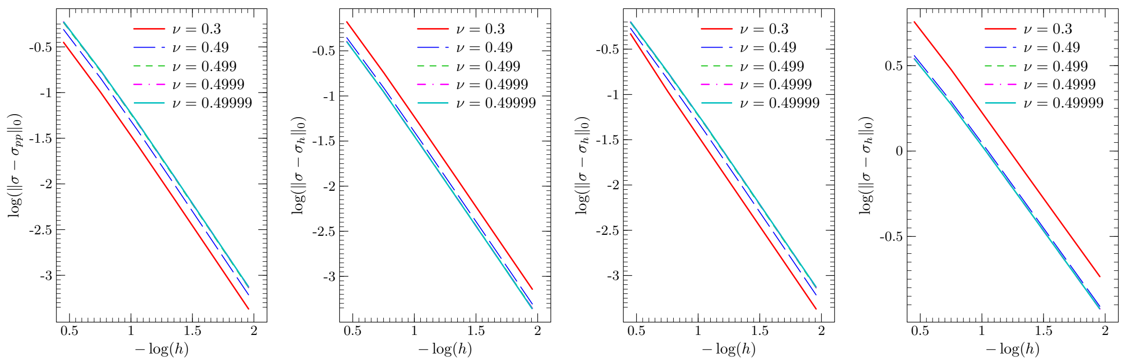
5.2 Convergence rates for the stress tensor

Figures 2 and 3 present the convergence rates in L^2 and $H(\text{div})$ norm for the stress tensor approximations when calculated using the constitutive equation, σ_h, or by the post-processing technique, σ_{pp}, as presented in Section 4.1, respectively. In this study, we consider the parameters $\beta = 20$, $\mu = 0.3, 0.49, 0.499, 0.4999, 0.49999$ (nearly incompressible) and the local stabilization parameter $\delta = 1$ for all elements $K \in \mathcal{T}_h$. We use triangular elements $P_k - p_k$ with $(k = 1, 2)$, adopting same order of interpolations k for displacement field and the Lagrange multiplier. For the stress field, we use $k+1$. As expected for $k = 1$, σ_h, calculated from the constitutive equation does not converge in H(div) norm and for $k = 2$ converges with order $\mathcal{O}(h^1)$. Improved rates of convergence $\mathcal{O}(h^1)$, for $k = 1$, $\mathcal{O}(h^2)$, for $k = 2$ are obtained for σ_{pp}, calculated by the proposed post-processing technique. Convergence rates are observed for all approximations σ_{pp}, independent of ν, using triangular elements. Same accuracies are obtained for $\nu = 0.49,\ 0.499,\ 0.4999,\ 0.49999$.



(a) Stress $\|\sigma_{pp}\|_0$ (b) Stress $\|\sigma_h\|_0$ (c) Stress $\|\sigma_{pp}\|_{div}$ (d) Stress $\|\sigma_h\|_{div}$

Figure 2: h-convergence for σ_{pp} and σ_h aproximations, in L_2 and $H(\text{div})$ norm using $k+1 = 2$ and $\mu = 0.3, 0.49, 0.499, 0.4999, 0.49999$.



(a) Stress $\|\sigma_{pp}\|_0$ (b) Stress $\|\sigma_h\|_0$ (c) Stress $\|\sigma_{pp}\|_{div}$ (d) Stress $\|\sigma_h\|_{div}$

Figure 3: h-Convergence for σ_{pp} and σ_h approximations in L_2 and H_div norm using $k+1 = 3$ and $\mu = 0.3, 0.49, 0.499, 0.4999, 0.49999$.

6 Concluding Remarks

We have presented a Stabilized Hybrid Discontinuous Galerkin finite element formulation for incompressible linear elasticity problems, based in [4]. The method was developed from a primal hybrid formulation, with the Lagrange multiplier identified with the trace of the displacement field on the edges of the elements, leading to a set of local problems defined at the element level and a global problem in the multiplier only. The formulation preserves the main properties of the corresponding DG method, but with reduced computational complexity. Numerical experiments confirm the locking free ability of the method for nearly incompressible problems. Accurate approximations and optimal rates of convergence in L_2 norm are obtained for triangular elements on uniform meshes.

A local post-processing based on residual forms of the constitutive and equilibrium equations at the element level, and using the multiplier as a kinematic boundary condition, is presented to recover stress approximations with improved stability, accuracy and robustness. Numerical experiments show that, compared to the standard post-processing based on the constitutive law, it retrieves optimal convergence rates in $H(\mathrm{div})$ norm and improves accuracy using triangular elements.

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