Proceeding Series of the Brazilian Society of Applied and Computational Mathematics, Vol. 5, N. 1, 2017.

Trabalho apresentado no CNMAC, Gramado - RS, 2016.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

Modeling the Parametric Dependence in a Linear Circuit by Experimental Measurements

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Abstract. This work presents an analytical characterization of the parametric dependence of variables of resistive circuits by few data samples. Cramer's solution of a linear system is alternatively formulated here as a rational function of polynomials of the parameters under study. Coefficients for these polynomials are calculated from few measures of system variables and known parameters. Discontinuities induced by measurement noises are fixed by regularization methods. Results are shown for both circuit simulations and experimental measurements. Ongoing work includes the application of the proposed approach in the synthesis of systems.

Keywords. Circuit analysis, Measurement-based method, Parametric rational function, Regularization

1 Introduction

Understanding the behavior of systems implies full knowledge of relationships between all its inputs and outputs. In mathematical modeling the aim is to describe within a certain level of detail the structure and interaction of variables and parameters in a system [5]. Identification techniques are used to approximate under *a priori* knowledge (e.g. the structure of the model) system parameters, being cumbersome for complex processes that are subjected to changes. A recent approach based on a reduced set of measurements have been proposed in [1] to estimate the relationship between system outputs and parametric rational functions by a linear approximation. This approach has been applied to obtain models of alternating current (AC) and direct current (DC) circuits [7], mechanical

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systems [6] and to design linear controllers [1]. In general terms, this approach can be classified as an inverse problem as it tries to find the rule (i.e. the system) producing the existing data [4]. There are strong ill-conditioning problems associated to inverse problems reported in the literature [3, 4, 8], which reduces the success of the problem of finding stable solutions under real noisy data. This paper applies the so-called measurement-based method for characterizing the parametric dependence of variables of a resistive circuit, showing practical aspects that should be taking into account in order to get accurate solutions under real experimental data.

2 Mathematical preliminaries

Let us start by considering a rational function of the form

$$y = f(p) = \frac{\beta_0 + \beta_1 p}{\alpha_0 + \alpha_1 p} \tag{1}$$

where $p \in \mathbb{R}$ is a parameter varied in a given interval and $\{\beta_i, \alpha_i\}$, $i = \{0, 1\}$ are unknown real constant weights. Then, a re-arrangement of (1) produces

$$\alpha_0 y(k) + \alpha_1 p(k) y(k) = \beta_0 + \beta_1 p(k), \forall k \in \mathbb{Z} \tag{2}$$

with $p(k)$ sampled values of p and $y(k)$ the corresponding outputs. In order to determine the unknown coefficients β_i and α_i, a system of equations can be constructed from (2). Given the rational form of (1), the linear system

$$\mathbf{G}\boldsymbol{\Phi} = \mathbf{0} \tag{3}$$

where

$$\mathbf{G} = \begin{bmatrix} y(1) & p(1)y(1) & -1 & -p(1) \\ y(2) & p(2)y(2) & -1 & -p(2) \\ y(3) & p(3)y(3) & -1 & -p(3) \\ y(4) & p(4)y(4) & -1 & -p(4) \end{bmatrix}, \quad \boldsymbol{\Phi} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \\ \beta_1 \end{bmatrix}$$

has nontrivial solutions and therefore, there must be a rank deficiency in matrix $\mathbf{G}$. Actually, $\text{rank}(\mathbf{G}) = 3$ and then the null-space of $\mathbf{G}$ has 1 vector. This introduces one-degree of freedom to the linear system (3) such that by selecting, for instance, a value for α_0 it can be alternatively represented by

$$\mathbf{M}\mathbf{z} = \mathbf{v} \tag{4}$$

with

$$\mathbf{M} = \begin{bmatrix} p(1)y(1) & -1 & -p(1) \\ p(2)y(2) & -1 & -p(2) \\ p(3)y(3) & -1 & -p(3) \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} \alpha_1 \\ \beta_0 \\ \beta_1 \end{bmatrix}, \quad \mathbf{v} = \alpha_0 \begin{bmatrix} -y(1) \\ -y(2) \\ -y(3) \end{bmatrix}$$

Solvability of (4) depends on the non-singularity of matrix $\mathbf{M}$. Then, for full-rank $\mathbf{M}$ a unique non-trivial solution $\mathbf{z}$ is related with a non-null α_0. The well-posedness of the system plays also an important role in the correctness of solutions, specially under noisy measurements. A problem is called well-posed in the sense of Hadamard [4] if there exists a unique solution which depends continuously on the data and parameters. Otherwise, we are facing an ill-posed problem which should be treated with regularization tools [8] in order to reduce the influence of noise on the calculations. Roughly speaking, regularizing an ill-posed problem consists in modifying its original formulation in such a way that approximate solutions can be constructed, converging continuously to the original (unperturbed) one [8]. There are many regularization methods reported in the literature, fitted to almost any kind of application problem. A basic approach was proposed by A. N. Tikhonov [4] in order to approximate the solution $\mathbf{z}$ of the original, unperturbed, linear system (4), stated in terms of the following minimization problem

$$\mathbf{z}_\lambda = \arg\min \left\{ \left\| \tilde{\mathbf{M}}\tilde{\mathbf{z}} - \tilde{\mathbf{v}} \right\|^2 + \lambda \left\| \tilde{\mathbf{z}} \right\|^2 \right\} \tag{5}$$

where $\mathbf{z}_\lambda$ is the approximated solution, $\tilde{\mathbf{z}}$ is noisy and λ is the regularization parameter [4]. The L-curve and other related methods for selection of λ can be found in [2] and references therein. The interested reader is referred to [8] for a deeper study on regularization methods and selection of regularization parameters.

3 A circuit output described by a rational function

The schematic diagram for a linear resistive circuit with only independent sources is shown in Fig. 1.

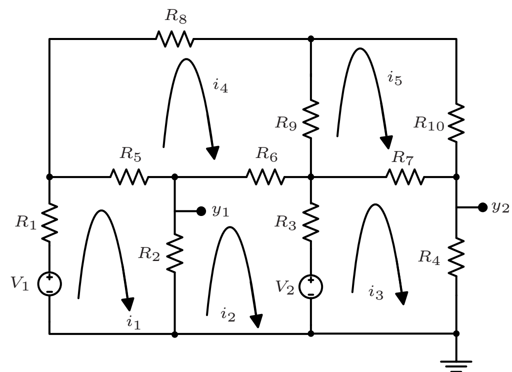


Figure 1: Resistive circuit for analysis of the measurement-based parametric predictions.

Using the Kirchhoff's voltage law (KVL), the output voltages can be described in the matrix form

$$\mathbf{A}\mathbf{x} = \mathbf{b}_1 V_1 + \mathbf{b}_2 V_2 \tag{6}$$

where

$$\mathbf{A}=\begin{bmatrix} R_1+R_2+R_5 & -R_2 & 0 & -R_5 & 0 & 0 & 0 \\ -R_2 & R_2+R_3+R_6 & -R_3 & -R_6 & 0 & 0 & 0 \\ 0 & -R_3 & R_3+R_4+R_7 & 0 & -R_7 & 0 & 0 \\ -R_5 & -R_6 & 0 & R_5+R_6+R_8+R_9 & -R_9 & 0 & 0 \\ 0 & 0 & -R_7 & -R_9 & R_7+R_9+R_{10} & 0 & 0 \\ -R_2 & R_2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -R_4 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}=\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ y_1 \\ y_2 \end{bmatrix} \quad \mathbf{b}_1=\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{b}_2=\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

with R_j, $j=1,2,...10$ and V_ℓ, $\ell=1,2$, known values. Then, it is possible to solve (6) for any x_n of $\mathbf{x}$ by applying the Cramer's rule as follows

$$x_n=\frac{|\mathbf{T}_{1,n}|\,V_1+|\mathbf{T}_{2,n}|\,V_2}{|\mathbf{A}|}=\frac{|\mathbf{T}_{1,n}|}{|\mathbf{A}|}V_1+\frac{|\mathbf{T}_{2,n}|}{|\mathbf{A}|}V_2 \tag{7}$$

with $|\mathbf{T}_{\ell,n}|$ denoting the determinant of $\mathbf{A}$ after replacing its n-th column by $\mathbf{b}_\ell$. Solution x_n in (7) can be alternatively computed using a recent approach proposed in [1] to estimate the relationship between variables and parameters in linear systems, based on measurement data. In particular, this approach states that under the assumption of rank m for a matrix with respect to a given parameter, the determinant of this matrix can be represented by an equivalent polynomial of order m with respect to that parameter [1]. The solution x_n can be easily shown to be [1]

$$x_n\,(p)=\frac{\beta_1(p)}{\alpha_1(p)}V_1+\frac{\beta_2(p)}{\alpha_2(p)}V_2 \tag{8}$$

where the unknown coefficients for the polynomials

$$\begin{aligned}\beta_\ell(p) &= \beta_{0\ell}+p\beta_{1\ell}\\ \alpha_\ell(p) &= \alpha_{0\ell}+p\alpha_{1\ell}\end{aligned}$$

with $\ell=1,2$, can be determined after proper selection of samples for the pair (x_n,p) and by assuming some normalization base, as shown previously in Section 2.

4 Parametric analysis

Simulations in OrCAD® and real measurements in laboratory were conducted in the circuit of Fig. 1 with the components given in Table 1. To illustrate the experimental setup, Fig. 2 depicts a black box scheme configured to find the input-output relationship

DOI: 10.5540/03.2017.005.01.0396

under variations of R_{10}. The voltage y_1 is described as a monotonic function of the resistance R_{10}. Also, the analytical predictions of the parametric function (8) for $n = 6$ and $p = R_{10}$, are obtained and the coefficients calculated from experimental data as presented in Table 2.

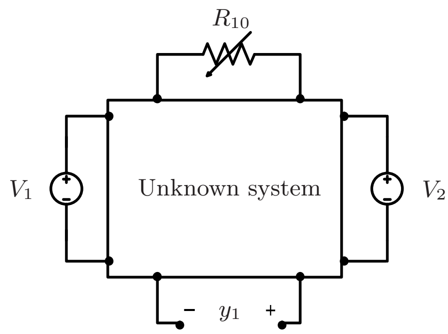


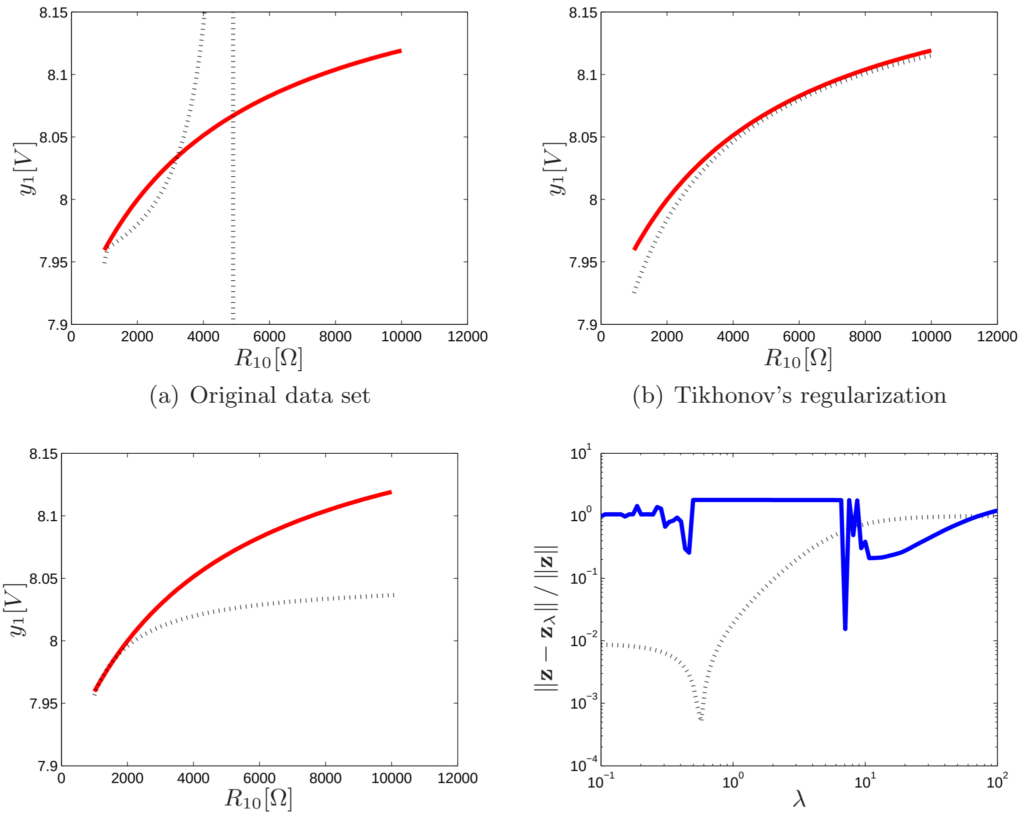
Figure 2: Experimental setup for analysis of one-parameter parametric dependence, showing the parameter varied, system inputs and output.

Table 1: Voltages and resistances used for circuit analysis.

V_1	V_2	R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}
18V	15V	$1k\Omega$	$4.7k\Omega$	$3.9k\Omega$	$5.6k\Omega$	$6.8k\Omega$	$5k\Omega$	$4.7k\Omega$	$5.6k\Omega$	$2.2k\Omega$	$\in [1k\Omega, 10k\Omega]$

The negative signs of coefficients α_{11} and α_{12} (see Table 2) create a discontinuity in the prediction as depicted in Fig. 3(a). This particular situation shows the ill-posedness of system (4) under measurement data. Results for the rational function after regularization using Tikhonov's and Maximum entropy (for comparison purposes) algorithms are presented in Figs. 3(b)-(c), recovering the smoothness of the predicted values of y_1 under variation of R_{10}. The regularization algorithms employed here were run in MATLAB and are described in [3]. The regularization parameters were selected by analyzing the regularization error shown in Fig. 3(d). Then, for the Tikhonov's regularization the best λ was found to be close to 0.7 and for the maximum entropy case it was found to be close to 10. Here it is important to remark that the value of λ is a compromise between sensitivity and precision of the solution. As can be seen, the smaller error in the prediction over a wider range of parameter values is obtained under Thikhonov's regulation. Nevertheless, there is also a remarkable drift for $R_{10} < 3k\Omega$ that can be explained by the effort needed to correct the original discontinuity around this value. The maximum entropy approach shows a good fit for lower values but bad predictions for higher values in the parameter range. This suggests a further study on dependence of predictions with respect to the sample set selected to evaluate (4). The coefficients calculated after Tikhonov's regularization are included in Table 2, noticing that the regularization removes discontinuities by preserving the positive sign of the denominator coefficients in (8).

DOI: 10.5540/03.2017.005.01.0396



(a) Original data set (b) Tikhonov's regularization

(c) Maximum entropy regularization (d) Regularization error

Figure 3: (a)-(c): Parametric dependence of y_1 with respect to variations in R_{10} obtained by simulations in OrCAD® (solid line) and by real measurements (dotted line) for both cases of original and regularized data sets. (d) Regularization error vs. λ: Tikhonov (dotted), Maximum entropy (solid).

Table 2: Coefficients of the one-parameter analytical function assuming $\alpha_{0\ell} = 1$, $\ell = 1, 2$.

Non-regularized		Regularized	
Coefficients	Value	Coefficients	Value
α_{11}	-0.200$\times 10^{-3}$	α_{11}	0.316$\times 10^{-3}$
β_{01}	0.3004	β_{01}	0.296
β_{11}	-0.595$\times 10^{-4}$	β_{11}	0.989$\times 10^{-4}$
α_{12}	-0.100$\times 10^{-2}$	α_{12}	0.114$\times 10^{-2}$
β_{02}	0.1693	β_{02}	0.1646
β_{12}	-0.169$\times 10^{-3}$	β_{12}	0.195$\times 10^{-3}$

5 Discussion and conclusion

Parametric dependence of output variables in a resistive circuit has been studied by a rational function of polynomials of the parameter under study. The analytical expression has been evaluated after calculation of the coefficients, based on a few set of measurements of the output variables for known parameter values. This avoids the necessity of having *a priori* information about the structure of the system in order to obtain a valid input (parameter) to output (variable of interest) relationship. Regularization of data was necessary to avoid discontinuities in predictions induced by measurement noises. Results presented here can be extended to analyze simultaneous variation of two and more parameters. The simplicity of the calculations involved in the method and the use of few measurements shows the potential application of the measurement-based approach to perform analysis of systems.

Acknowledgements

This work was supported by the Brazilian National Council for Scientific and Technological Development (CNPq) under Ciência Sem Fronteiras Program grants 150228/2015-5, 150339/2016-0 and 400401/2013-4. The first author acknowledges the financial support of the São Paulo Research Foundation (FAPESP) under grant 2014/26095-0.

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DOI: 10.5540/03.2017.005.01.0396