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Elastoplastic Modeling Using The Successive Linear Approxitamtion Method

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Abstract. The goal of this work is to model elastoplastic materials. More precisely, it will be considered metals with von Mises criterion. For this model the successive linear approximation method and the small plastic deformation theory will be considered. The finite element method will be employed for numeric solution for each force increment.

Keywords. successive linear approximation, plasticity, von Mises, finite elements

1 Introduction

Large deformation is a subject of interest in the field of computational mechanics. Most of this interest lies in nonlinear problems which can be divided in two types: material nonlinearity and geometrical nonlinearity.

The computational methods more typically used, and that are very well documented, are based on the theories of [7], [6] and [9]. These methods use Newton's method to solve the nonlinear system obtained after the discretization of the problem.

Incremental methods were proposed over the years to solve large deformation through small deformations, like in [4] and [1]. In [2] was proposed a incremental method for elastic material called successive linear approximation method. In [3] this method was extend to the viscoelastic material case.

Despite apparent similarities between the successive linear approximation(SLA) method with other incremental methods, SLA uses a formulation that considers the current state of the body as the reference configuration, while the others are usually written in a Lagrangean formulation.

The objective of this work is to verify that the SLA method can be adapted to simulate elastoplastic small deformation. To do so, we will use the classical theory of plasticity for small deformation.

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2 Successive Linear Approximation

Boundary Value Problems (BVP), in general, are formulated in referential (Lagrangian) or spacial (Eulerian) coordinates. Alternatively, the *Successive Linear Approximation Method (SLA)* uses the *relative-descriptional formulation.* In this formulation, the BVP are formulated in coordinates concerning the configuration of the current time.

In this method, the constitutive equations are calculated at each state, with the reference configuration updated for each time step. The new reference configuration is the current configuration of the body. Assuming that in each time step occurs a small deformation, the constitutive equations are linearized. In this way, the SLA method solves a nonlinear problem by linear increments for each time step, i.e., a large deformation can be calculated by increments of small deformations.

2.1 Updated reference configuration

Let $\mathcal{B}$ be an elastic body, with κ_0 being the preferred reference configuration. Consider

$$\mathbf{x} = \chi(X,t), \qquad X \in \kappa_0((B)) = \mathcal{B}_0 \tag{1}$$

the *deformation* of the body $\mathcal{B}_0$ at time t. The *Cauchy stress tensor* is defined by the constitutive equation

$$T(X,t) = \mathcal{F}_{\kappa_0}(F) \tag{2}$$

where $F(X,t)$ is the *gradient deformation* and $\mathcal{F}_{\kappa_0}(F)$ is the constitutive function of the elastic material. In general, this function is a nonlinear function of F.

Let κ_t be a deformed configuration at time t, $\mathcal{B}_t = \kappa_t(\mathcal{B})$. In this way, we can define the gradient deformation with respect to the configuration κ_0,

$$F(X,t) = \nabla_X \chi(X,t). \tag{3}$$

Now, consider a time $\tau > t$ and κ_τ a deformed configuration in this time, $\mathcal{B}_\tau = \kappa_\tau(\mathcal{B})$. In time τ, the deformation relative to κ_0 is given by $\xi = \chi(X,\tau)$. Therefore, it is possible to define, respectively, in the current configuration, κ_t, the *relative deformation* and the *relative displacement* from κ_t to κ_τ, using the function $\chi_\tau : \mathcal{B}_t \to \mathcal{B}_\tau$

$$\chi_\tau(\mathbf{x},\tau) := \chi(X,\tau), \qquad \mathbf{x} \in \mathcal{B}_t \tag{4}$$

$$u_t(\mathbf{x},\tau) := \xi - \mathbf{x} = \chi_\tau(\mathbf{x},\tau) - \mathbf{x}, \qquad \mathbf{x} \in \mathcal{B}_t. \tag{5}$$

Taking the gradient relative to $\mathbf{x}$ in both sides of equation (5), we have

$$H_t(x,\tau) \;=\; F_t(x,\tau) - I \tag{6}$$

where H_t and F_t are called, respectively, *relative displacement gradient* and *relative deformation gradient*. I is the *identity tensor*.

Now, calculating the gradient of equation (5) relative to X, we have

$$F(X,\tau) \;=\; (I + H_t(\mathbf{x},\tau))F(X,t). \tag{7}$$

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The diagram below represents the whole situation so far:

$$\begin{tikzcd}
 & X \in \mathcal{B}_0 \arrow[dl, "F(t)"'] \arrow[dr, "F(\tau)=(I+H)F(t)"] & \\
\mathbf{x} \in \mathcal{B}_t \arrow[rr, "F_t(\tau)=I+H", "\xi=\mathbf{x}+u_t(\xi,t)"'] & & \xi \in \mathcal{B}_\tau
\end{tikzcd}$$

With these concepts established, it is possible to define a function $f_t(\mathbf{x}, \tau)$ over the domain $\mathcal{B}_t \times \mathbb{R}$, at time τ concerning to the current configuration, as if this function were seen in the instant τ from a observer attached to the body in its movement at the current time t. This features what we call earlier *relative-descriptional formulation.*

2.2 Linearized constitutive equation

By the literature it is known that the *Hooke's Law* don't satisfies the principle of material frame-indifference. Therefore it can only be regarded as a approximation for small deformations. For large deformations we consider that $\mathcal{F}_{\kappa_0}(F)$ is a nonlinear functional in relation to F.

Let $\tau = t + \Delta t$, where we assume that Δt is small enough so that the displacement gradient is small ($H \ll 1$), i.e., $H(\tau) = H_t(\mathbf{x}, \tau)$. With this, using equation (6) and equation (7), we can conclude

$$F(\tau) - F(t) = H(\tau)F(t) \qquad F_t(\tau) = I + H(\tau) \tag{8}$$

Now, using *Taylor* to linearize equation (2) relative to the current configuration κ_t, we have:

$$T(\tau) \quad = \quad T(t) + \nabla_F \mathcal{F}_{\kappa_0}(F(t))[F(\tau) - F(t)] = T(t) + L(F(t))[H(\tau)] \tag{9}$$

where $L(F(t))$ is the fourth order elasticity tensor relative to the reference configuration κ_t.

Since we have the Cauchy stress tensor, we can also define the first *Piola-Kirchhoff* stress tensor, $T_{\kappa_t}(\tau)$, at time τ relative to the current configuration κ_t,

$$T_{\kappa_t}(\tau) = T(t) + (trH)T(t) - T(t)H^T + L(F)[H] + o(2) = T(t) + \mathcal{L}(F, T)[H]. \tag{10}$$

where $\mathcal{L}(F, T)[H]$ is the forth order elasticity tensor for the Piola-Kirchhoff stress tensor.

2.3 Numerical method for large deformation

It is possible to solve numerically large deformations problems using the same strategy that the Euler's method for differential equations do. Consider the discrete time axis $\cdots < t_{n-1} < t_n < t_{n+1} < \cdots$ where $t_{n+1} = t_n + \Delta t$. Assume that Δt is small enough. Consider κ_{t_n} is the body configuration at time t_n and $\mathbf{x}_n = \chi(X, t_n) \in \mathcal{B}_{t_n}$ with $X \in \mathcal{B}_{t_0}$

Let $F(\mathbf{x}_n, t_n)$ and $T(\mathbf{x}_n, t_n)$ be, respectively, the gradient deformation and the Cauchy stress relative to the preferred reference configuration κ_0, assumed to be known. If in any way we calculate the relative displacement $u_{t_n}(\mathbf{x}_n, t_n)$, it allows us the update the new reference configuration, $\kappa_{t_{n+1}}$, relative to the next step by using

$$\mathbf{x}_{n+1} = \chi(X, t_{n+1}) = \mathbf{x}_n + u_t(\mathbf{x}_n, t_{n+1}) \tag{11}$$

while the deformation gradient and the Cauchy stress, relative to the preferred configuration κ_0 can be determined at instant t_{n+1} by

$$F(\mathbf{x}_{n+1}, t_{n+1}) = (I + H_{t_n}(\mathbf{x}_n, t_{n+1}))F(\mathbf{x}_n, t_n) \tag{12}$$

$$T(\mathbf{x}_{n+1}, t_{n+1}) = T(\mathbf{x}_n, t_n) + L(F(\mathbf{x}_n, t_n))[H_{t_n}(\mathbf{x}_n, t_{n+1})] \tag{13}$$

Therefore, after updating the boundary data and the eventual body forces acting on the body, we repeat the cycle from the updated reference configuration $\kappa_{t_{n+1}}$. This process is known as successive linear approximation.

2.4 Linearized boundary value problem

Let $\Omega = \kappa_t(\mathcal{B}) \in \mathbb{R}^2$ be the region occupied by the body at the current configuration κ_t at time t. Let $\partial\Omega = \Gamma_1 \bigcup \Gamma_2$ be the boundary of the body and $\mathbf{n}_{\kappa_\mathbf{t}}$ be the exterior unit normal to $\partial\Omega$. Let the relative displacement vector from κ_t to κ_τ be $\mathbf{u}(\tau) = u_t(\mathbf{x}, t+\Delta t) = \chi(X, \tau) - \chi(\mathbf{x}, t)$ and the displacement gradient be, $H(\tau) = \nabla_\mathbf{x}\mathbf{u}(\tau)$.

Consider a BVP of an elastic body in equilibrium without external body forces, at time $\tau = t + \Delta t$ relative to the current configuration at time t given by

$$\left\{\begin{array}{rcll} -divT_{\kappa_t}(\mathbf{x}, \tau) & = & 0 & \textbf{in} \quad \Omega \\ T_{\kappa_t}(\mathbf{x}, \tau)\mathbf{n}_{\kappa_t} & = & \mathbf{f} & \textbf{in} \quad \Gamma_1 \\ \mathbf{u}(\mathbf{x}, \tau) & = & \mathbf{g} & \textbf{in} \quad \Gamma_2 \end{array}\right. \tag{14}$$

where $\mathbf{f}$ is the prescribed surface traction, $\mathbf{g}$ is the prescribed displacement surface and $\mathbf{u}$ is the displacement vector from κ_t to κ_τ. In equation (14) T_{κ_t} is the Piola-Kirchhoff stress tensor at time τ relative to configuration κ_t at present time t. Replacing equation (10) in equation (14) the equilibrium equation becomes

$$-div(\mathcal{L}(F, T)[H(\tau)]) = div(T(t)) \tag{15}$$

In this problem, the time t is supposed to be known, i.e., $F(t)$ and $T(t)$ are known. Therefore, equation (15) is a linear partial differential equation in time τ, and the system in equation (14) is a linearized BVP for the determination of the relative displacement vector $\mathbf{u}(\mathbf{x}, \tau)$, as follows

$$\left\{\begin{array}{rcll} -div(\mathcal{L}(F, T)[\nabla_\mathbf{x}\mathbf{u}(\mathbf{x}, \tau)]) & = & div(T(t)) & \textbf{in} \quad \Omega \\ (\mathcal{L}(F, T)[\nabla_\mathbf{x}\mathbf{u}(\mathbf{x}, \tau)])\mathbf{n}_\kappa & = & \mathbf{f} - T(t)\mathbf{n}_\kappa & \textbf{in} \quad \Gamma_1 \\ \mathbf{u}(\mathbf{x}, \tau) & = & \mathbf{g} & \textbf{in} \quad \Gamma_2 \end{array}\right. \tag{16}$$

The linearization that we made does not depend on the incremental loading, like in other methods [4], [1]. At every time step, the idea is to formulate a BVP as in equation (16) and superposing this small deformations in large deformation.

3 Elastoplastic successive linear approximation

The successive linear approximation method is based on the fact that in each force increment a small deformation occurs. The premise which we start is that in this increment, if the plastic deformation takes place, the mathematical theory of plasticity for small deformation is valid.

The mathematical theory of plasticity for small deformation is based on Hooke's law, therefore, for the model that we will adopt the forth order elastic tensor will be the same that Hooke's law uses, i.e.,

$$L_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{17}$$

where λ and μ are the material parameters known as *Lamé's parameters*.

In this work we are only concerned with metals, and thus the *von Mises yield criterion* will be used. The relationship between stress and strain to describe material behavior when this is made up of both elastic and plastic components that we use is the classical:

$$dT = L_{ep} d\varepsilon, \qquad L_{ep} = L - \frac{\mathbf{a} : L \otimes L : \mathbf{a}}{h' + \mathbf{a} : L : \mathbf{a}}, \qquad \mathbf{a} = \frac{\partial f}{\partial T} \tag{18}$$

where dT and $d\varepsilon$ are, respectively, the increment representation of the stress and strain tensors. L_{ep} is the elastoplastic tensor of material properties. h' is a hardening function and f is the yielding function.

3.1 Application: Thick cylinder subjected to a gradually increasing internal pressure

This example was studied in [5], [8] and others. In this case the condition of plane strain is assumed. The material properties are: $E = 2.1 \times 10^4 dN/mm^2$, $\nu = 0.3$, $\sigma_y = 24 dN/mm^2$ and $h' = 0$. The finite element mesh used was composed by 120 linear elements of Q_4 type. This mesh and the boundary condition are shown in Figure (1). The linear yield function used is defined as

$$\overline{T} = \sigma_y + h' d\overline{\varepsilon}^p \tag{19}$$

where $\overline{T}$ is the effective stress and $d\overline{\varepsilon}^p$ is the effective plastic strain.

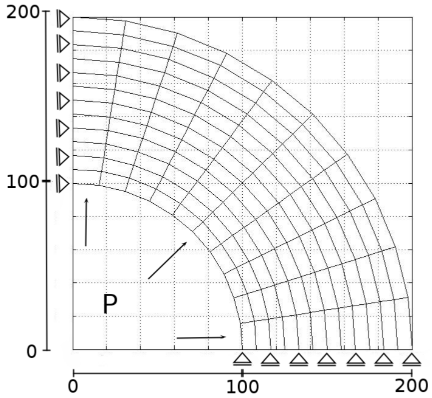


Figure 1: Mesh and boundary conditions

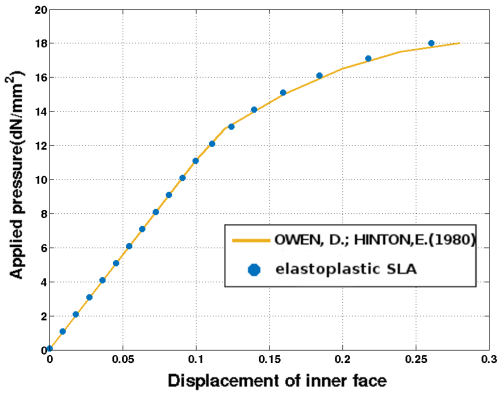
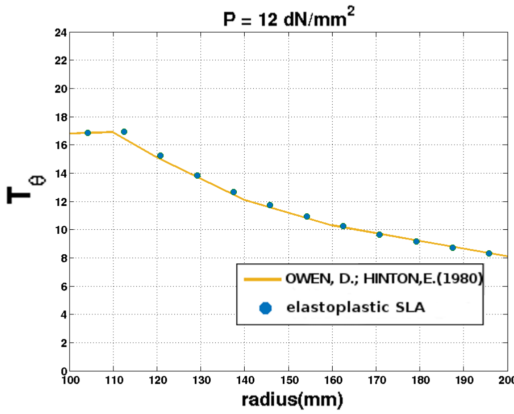


Figure 2: Displacement of inner face × applied pressure.

Figure (2) compares the theoretical and numerical solution of the relation between the increasing of the applied pressure with the displacement of the inner face. In Figure (3) we present the results of the distribution of principal stress T_θ along the radial axis of the vessel at various pressure values.



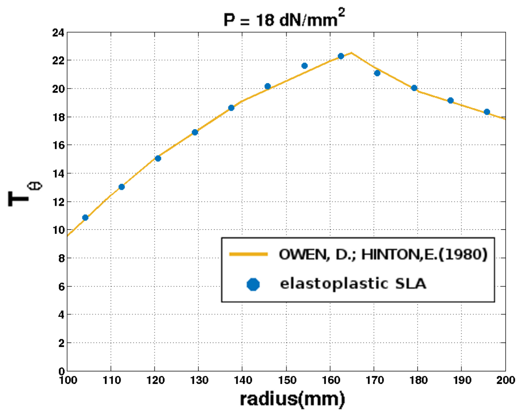


Figure 3: Principal stress distribution $T_\theta = \frac{T_1+T_2}{2} + \sqrt{(\frac{(T_1-T_2)^2}{4} + T_{12}^2)}$ at pressure values $P = 12dN/mm^2$ and $P = 18dN/mm^2$.

In this simulation was used 180 steps of the SLA method with $\Delta P = 0.1$. The results are compared with the theoretical solutions shown in [5] and they agree with the reference.

4 Conclusion

In this work the SLA method for elastoplastic material was proposed. For the example in small deformations the method had an excellent behavior. Therefore, results were satisfactory, showing that the elastoplastic SLA has a similar behavior with the articles considered.

The advantages of using SLA methods are that there is no trouble in impose the boundary conditions and update the material configuration. Another important topic is that with SLA method we can solve the material nonlinearity with the theory of plasticity for small deformations.

For the future works, will be interesting to implement large defomation of elastoplastics material with this model. More yielding criteria for metals and for soils have to be implemented to prove the strenght of this model.

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