STABILITY ANALYSIS OF THE FOURTH-ORDER CUBIC MEMRISTOR OSCILLATOR

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Abstract— In this paper we present the stability analysis of one memristor oscillator mathematical model, given by four-dimensional five-parameter cubic system of ordinary differential equations. Through the Routh-Hurwitz stability criteria, we prove that the equilibrium points are stable for a fixed set of parameters.

Keywords— Memristor, Stability, Routh-Hurwitz Criterion.

Resumo— Neste artigo apresentamos a análise da estabilidade de um modelo matemático dado por um oscilador envolvendo o memristor, representado por um sistema de equações diferenciais ordinárias de quarta ordem com cinco parâmetros e uma função cúbica. Através do Critério de Estabilidade de Routh-Hurwitz, provamos que os pontos de equilíbrio são estáveis para um conjunto fixo de parâmetros.

Palavras-chave— Memristor, Estabilidade, Critério de Routh-Hurwitz.

1 Introduction

In 2008, a team of scientists of Hewlett-Packard Company announced the fabrication of a memristor, short for memory resistor, that is a passive nonlinear two-terminal circuit element that has a functional relationship between the time integrals of current and voltage (Strukov et al. (2008)). The memristor is the fourth fundamental electronic element in addition to the resistor, inductor and capacitor. The existence of memristors was postulated theoretically by the scientist Leon Chua in 1971 (Chua (1971)).

In fact, a memristor is essentially a resistor with memory. So, for example, a computer created from memristive circuits can remember what has happened to it previously, and freeze that memory when the circuit is turned off (Hopkin (1978)). Then, the memristor promises to be very useful in the fields of nanoelectronics and computer logic, for example. The announcement of the physical construction of the memristor has attracted worldwide attention due to its great potential in applications.

In Itoh and Chua (2008), we can see the basic equations analysis of several types of nonlinear memristor oscillators, obtained by replacing Chuas's diode with memristors in some of the well studied Chua's circuits.

In this paper we propose a new memristor model, obtained by replacing the memductance function (a piecewise-linear function) in the model presented in Itoh and Chua (2008), with a quadratic positive definite function. Furthermore, we present a detailed stability analysis of this model, showing that the equilibrium points are stable, for a fixed set of parameter values.

2 A fourth-order memristor-based Chua oscillator

As described in Itoh and Chua (2008), the memristor is a passive two-terminal eletronic device described by a nonlinear constitutive relation between the device terminal voltage v and the terminal current i, given by

$$v = M(q)i, \quad \text{or} \quad i = W(\varphi)v.$$

The functions $M(q)$ and $W(\varphi)$, which are called *memresistance* and *memductance*, are defined by

$$M(q) = \frac{d\varphi(q)}{dq} \geq 0 \quad \text{and} \quad W(\varphi) = \frac{dq(\varphi)}{d\varphi} \geq 0,$$

and represent the slope of scalar functions $\varphi = \varphi(q)$ and $q = q(\varphi)$, respectively, named the *memristor constitutive relation*.

In Itoh and Chua (2008), the authors give several mathematical models for memristor oscillators by replacing Chua's diodes with memristors in some Chua's circuits. Here we consider the fourth-order cubic canonical memristor oscillator obtained from the canonical Chua's circuit (see Fig. 1). Removing a resistor from the circuit of Fig. 1, we obtain the fourth-order oscillator in Fig. 2.

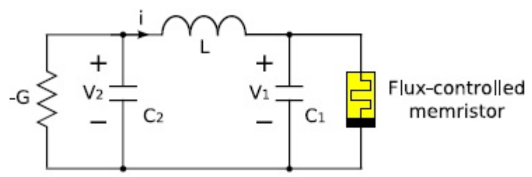

Figure 1: Canonical Chua's oscillator with a flux-controlled memristor. Reference: Itoh and Chua (2008).

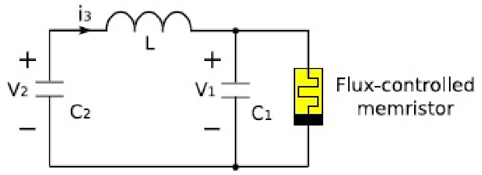

Figure 2: A fourth-order oscillator with a flux-controlled memristor. Reference: Itoh and Chua (2008).

We can represent the dynamics of the circuit represented in Fig. 2 by

$$\begin{cases} C_1\dfrac{dv_1}{dt} = i_3 - W(\varphi)v_1 \\ L\dfrac{di_3}{dt} = v_2 - v_1 \\ C_2\dfrac{dv_2}{dt} = -i_3 \\ \dfrac{d\varphi}{dt} = v_1 \end{cases}, \qquad (1)$$

where the memristor is characterized by a monotone-increasing cubic function given by

$$q(\varphi) = \varphi^3 + a\varphi^2 + b\varphi + c$$

from which we obtain the memductance $W(\varphi)$ as

$$W(\varphi) = \frac{dq(\varphi)}{d\varphi} = 3\varphi^2 + 2a\varphi + b.$$

We will also consider $a^2 - 3b \le 0$. So, we have $W(\varphi) \ge 0$, i.e., the memductance is positive definite and so the memristor is passive.

System (1) can be written as

$$\begin{cases} \dfrac{dx}{dt} = \alpha(y - W(w)x) \\ \dfrac{dy}{dt} = -\xi(x+z) \\ \dfrac{dz}{dt} = \beta y \\ \dfrac{dw}{dt} = x \end{cases}, \qquad (2)$$

where $x = v_1$, $y = i_3$, $z = -v_2$, $w = \varphi$, $\alpha = \dfrac{1}{C_1}$, $\beta = \dfrac{1}{C_2}$, $\xi = \dfrac{1}{L}$ and the cubic function $q(w)$ and quadratic function $W(w)$ are given by

$$\begin{array}{l} q(w) = w^3 + aw^2 + bw + c, \\ W(w) = 3w^2 + 2aw + b, \end{array}$$

respectively, where $a^2 - 3b \le 0$.

In Itoh and Chua (2008), the authors present numerical simulations and a stability study of system (2) using the functions $q(w)$ and $W(w)$, where

$$q(w) = bw + 0.5(a-b)(|w+1| - |w-1|),$$
$$W(w) = \begin{cases} a, & |w| < 1 \\ b, & |w| > 1 \end{cases},$$

respectively, and $a, b > 0$.

Here, we propose a new model, with the memductance given by a quadratic function. This type of consideration has already appeared in the literature (for example, Messias et al. (2010)).

3 Stability criteria

To analyse the stability of equilibrium points of a systems of ordinary differential equations we need to investigate whether all the zeros of a characteristic polynomial belong to the left half-plane, i.e., whether the polynomial is stable. There are several different solutions of this problem. In this paper we use the result due to Routh and Hurwitz, that may be found in Gantmacher (1959).

Theorem 1 (The Routh-Hurwitz Criterion) *All the zeros of real polynomial*

$$P(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1}z + a_n \ (a_0 > 0)$$

belong to the left half-plane if, and only if, $\Delta_i > 0$, $i = 1, \ldots, n$, *where*

$$\Delta_i = \begin{vmatrix} a_1 & a_3 & a_5 & \ldots & a_{2k-1} \\ a_0 & a_2 & a_4 & \ldots & a_{2k-2} \\ 0 & a_1 & a_3 & \ldots & a_{2k-3} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & a_k \end{vmatrix}$$

is the Hurwitz determinant of order i *and* $a_j = 0$ *for* $j > n$.

A real polynomial $P(z)$ whose coefficients satisfy the conditions $\Delta_i > 0$, $i = 1, \ldots, n$, is usually called a Hurwitz polynomial. The class of all such polynomials will be denoted by $\mathscr{H}$. More details can be found in Marden (1966).

4 Results

The equilibrium points of the system 2 are given by

$$E = \{(x,y,z,w) \in \mathbb{R}^4 \mid x = y = z = 0 \text{ and } w \in \mathbb{R}\}.$$

The Jacobian matrix J of system 2 at the equilibrium point $(0,0,0,w)$ is given by

$$J = \begin{bmatrix} -\alpha W(w) & \alpha & 0 & 0 \\ -\xi & 0 & -\xi & 0 \\ 0 & \beta & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

from which it follows that this equilibrium has the eigenvalues $\lambda_1 = 0$ and $\lambda_{2,3,4}$ given by the solutions of the cubic equation

$$P(\lambda) = \lambda^3 + \alpha W(w)\lambda^2 + (\alpha+\beta)\xi\lambda + \alpha\beta\xi W(w) = 0.$$

Using the Routh-Hurwitz criterion for the above polynomial, we have

$$\begin{aligned} \Delta_1 &= \alpha W(w); \\ \Delta_2 &= \begin{vmatrix} \alpha W(w) & \alpha\beta\xi W(w) \\ 1 & \xi(\alpha+\beta) \end{vmatrix} = \alpha^2\xi W(w); \\ \Delta_3 &= \begin{vmatrix} \alpha W(w) & \alpha\beta\xi W(w) & 0 \\ 1 & \xi(\alpha+\beta) & 0 \\ 0 & \alpha W(w) & \alpha\beta\xi W(w) \end{vmatrix} \\ &= \alpha^3\xi^2\beta W^2(w). \end{aligned}$$

Considering $a^2 - 3b < 0$, as $\alpha, \beta, \xi > 0$, then $\Delta_i > 0$, for $i = 1, 2, 3$. So, $P \in \mathscr{H}$ and the equilibrium state

$$E = \{(x, y, z, w) \in \mathbb{R}^4 \mid x = y = z = 0 \text{ and } w \in \mathbb{R}\}$$

is stable.

If $a^2 - 3b = 0$, $W(w)$ has one real zero of multiplicity two and for $w = -\dfrac{a}{3}$, $W(w) = 0$. For all $w \in \mathbb{R}$ such as $w \neq -\dfrac{a}{3}$, $W(w) > 0$. So, if $a^2 - 3b = 0$, we have:

- For all $w \in \mathbb{R}$ such as $w \neq -\dfrac{a}{3}$, $W(w) > 0$ and $\Delta_i > 0$, $i = 1, 2, 3$. So, $P \in \mathscr{H}$ and the equilibrium state

  $$E = \{(0, 0, 0, w) \in \mathbb{R}^4 \mid w \in \mathbb{R}\}$$

  is stable.

- If $w = -\dfrac{a}{3}$, $W(w) = 0$ and we could not use the Routh-Hurwitz Criterion. But, in this case, the zeros of the polynomial

  $$P(\lambda) = \lambda^3 + (\alpha + \beta)\xi\lambda$$

  are given by

  $$\lambda_2 = 0 \text{ and } \lambda_{3,4} = \pm i\sqrt{(\alpha + \beta)\xi}.$$

  Consequently, the point $\left(0, 0, 0, -\dfrac{a}{3}\right)$ is a centre (the stability is indeterminate).

So, we conclude that the equilibrium points of the system 2 are stable when the parameters vary in the set

$$\{(\alpha, \beta, \xi, a, b) \in \mathbb{R}^5 | \alpha, \beta, \xi > 0 \text{ and } a^2 - 3b < 0\}.$$

Furthermore, the equilibrium state

$$E = \left\{(0, 0, 0, w) \in \mathbb{R}^4 \mid w \in \mathbb{R} - \left\{-\frac{a}{3}\right\}\right\}$$

is stable when the parameters vary in the set

$$\{(\alpha, \beta, \xi, a, b) \in \mathbb{R}^5 | \alpha, \beta, \xi > 0 \text{ and } a^2 - 3b = 0\}.$$

Numerical Simulation

To illustrate the solutions of system 2 we create a program in software *Mathematica 8*. Fig. 3 shows the stability of the equilibria $(0, 0, 0, w)$ with the parameters $\alpha = 0.5$, $\beta = 4$, $\xi = 8$ and $a = b = 3$.

Figure 3: Solutions of system 2 with $\alpha = 0.5$, $\beta = 4$, $\xi = 8$ and $a = b = 3$ showing the stability of the equilibria $(0, 0, 0, w)$. Initial conditions: $x(0) = 0$, $y(0) = 0.11$, $z(0) = 0.11$ and $w(0) = 0$.

From the cursor in the top side of the Fig. 3, we can vary the parameters and obtain new solutions.

Conclusions

In this paper we propose a new fourth-order memristor model and we present its stability analysis, showing that the equilibrium points of the system 2 are stable when the parameters vary in the set

$$\{(\alpha, \beta, \xi, a, b) \in \mathbb{R}^5 | \alpha, \beta, \xi > 0 \text{ and } a^2 - 3b < 0\}$$

and proving that the equilibrium state

$$E = \left\{(0, 0, 0, w) \in \mathbb{R}^4 \mid w \in \mathbb{R} - \left\{-\frac{a}{3}\right\}\right\}$$

is stable when the parameters vary in the set

$$\{(\alpha, \beta, \xi, a, b) \in \mathbb{R}^5 | \alpha, \beta, \xi > 0 \text{ and } a^2 - 3b = 0\}.$$

Our future studies will be focused on the case that some parameters of system 2 are negative.

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