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On the Graph Laplacian for Spectral Image Segmentation and Energy Minimization on Graphs

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Abstract. Image segmentation is an indispensable tool to enhance the ability of computer systems to perform elementary cognitive tasks such as *recognition* and *tracking*. In particular, graph-based algorithms have gained a lot of attention lately, specially due to their good performance in clustering complex images and easy usability. However, most traditional approaches rely on sophisticated mathematical tools whose effectiveness strongly depends on how good the boundaries reflect the partitions of the image. In fact, sharp adherence to the contours of image segments, uniqueness of solution, high computational burden, and extensive user intervention are some of the weaknesses of most representative techniques. In this work we proposed two novel graph-based image segmentation techniques that sort out the issues discussed above. The proposed methods rely on Laplace operators, spectral graph theory, and optimization approaches towards enabling highly accurate segmentation tools which demand a small amount of user involvement while still being mathematically easy-to-handle and computationally efficient. The effectiveness of our segmentation algorithms is attested by a comprehensive set of comparisons against state-of-the-art methods. As additional contribution, we have also proposed two new techniques for image inpainting and photo colorization, both of which rely on the accuracy of our segmentation apparatus.

Keywords. graph laplacian, clustering, spectral theory, optimization, image processing.

1 Introduction

Segmentation is the key task for an enormous quantity of computer vision problems. A typical procedure in image segmentation is to interpret an image as a graph, which enables the use of powerful mathematical tools such as Laplace operators and spectral graph theory in the context of segmentation. Moreover, the flexibility introduced by a graph representation as to pixel connectivity and edge weighting greatly increases the capability of segmentation algorithms to distinguish patterns, structures, and shapes. However, outperforming human skills in terms of pattern recognition is a challenging task. Therefore, *semi-supervised segmentation* methods have become a trend by combining the human ability for patter recognition with the solid mathematical foundation of graph theory [1].

In this context, the use of interact mechanisms to properly settle Laplacian-based operators on image graph representations have proven to be an effective alternative [2, 3].

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The segmentation is accomplished by either optimizing a cost function [4, 5] or by solving a spectral-cut problem [2, 6]. Despite their pliability and powerful, most state-of-the-art methods often produce segmentations with low adherence to the contours of the image segments, failing to capture fine details and thus producing a low quality result [7]. Moreover, most existing techniques rely on sophisticated optimization resources that require high processing effort while being sensitive to edge weights and user intervention.

Contributions.

In this work we proposed two novel user-assisted image segmentation techniques that address the issues discussed above. The proposed algorithms rely on Laplace operators, spectral graph theory, and optimization tools towards reaching highly fitting on object boundaries which demand a reduced amount of user involvement while still being mathematically easy to solve and computationally efficient. While most of our research has been focused on the particular problem of segmentation, we developed as side results new methods for the problem of *inpainting* [8] and *photo colorization* [9], both of which derived from the previous segmentation methodologies combined with PDE-based approaches. The list below provides the main publications originated during the development of the PhD work:

Contributions in Graph Clustering & PDE-based Models for Images: [2,5,8–14].
Contributions in Optimization & Multidimensional Data Analysis: [15–18].

2 Spectral Image Segmentation

Spectral graph theory [1] has been the basic tool for the so-called spectral cut methodology [6, 19], which exploits the eigenstructure of an image affinity graph so as to perform clustering. Among the vast amount of techniques inspired in spectral cuts, three approaches have gained popularity in recent years, being widely used as source of segmentations in many practical applications: *Spectral and Normalized Cuts* [6, 20], *Multiscale Spectral Segmentation* [19, 21, 22], and *Random Walker Segmentation* [4]. Despite their effectiveness, those approaches typically present some weaknesses that must be observed when performing segmentation. For example, the accuracy in detecting the boundaries between image regions is highly dependent on the weights assigned to the edges of the graph [6, 19]. Another important issue involving spectral cuts is the numerical cost, as computing eigenstructures of a graph is a very time consuming task [22].

2.1 Spectral Segmentation via Cartoon-Texture Decomposition

In this section we brief describe the image segmentation technique proposed in [2, 11], which relies on spectral cuts but address the issues raised above. The proposed approach holds attractive properties such as awareness to noise and texture, accuracy in detecting image edges, low computational cost and it operates with a reduced number of human intervention. Our approach comprises four main steps, as presented below.

Cartoon-Texture Decomposition. It separates the target image $\mathcal{I}$ into two disjoint images, $\mathcal{C}$ and $\mathcal{T}$, so that $\mathcal{I} = \mathcal{C} + \mathcal{T}$. The cartoon component $\mathcal{C}$ holds the geometric structures and smooth-parts of $\mathcal{I}$ while the texture component, $\mathcal{T}$, contains textures and noise.

Similar to [23], where a functional minimization problem was formulated through a system of partial differential equations, both cartoon and texture components are computed by solving the following system of equations, where $\mathcal{T}=\mathrm{div}(g_1, g_2)$:

$$\begin{cases} \mathcal{C} = \mathcal{I} - \partial_x g_1 - \partial_y g_2 + \dfrac{1}{2\lambda}\,\mathrm{div}\left(\dfrac{\nabla\mathcal{C}}{|\nabla\mathcal{C}|}\right) \\ \mu\dfrac{g_1}{\sqrt{g_1^2+g_2^2}} = 2\lambda\left[\dfrac{\partial}{\partial x}(\mathcal{C}-\mathcal{I}) + \partial^2_{xx}g_1 + \partial^2_{xy}g_2\right] \\ \mu\dfrac{g_2}{\sqrt{g_1^2+g_2^2}} = 2\lambda\left[\dfrac{\partial}{\partial y}(\mathcal{C}-\mathcal{I}) + \partial^2_{xy}g_1 + \partial^2_{yy}g_2\right] \end{cases}.$$

Affinity Graph Construction. This step consists in building the similarity graph G by associating each pixel from $\mathcal{C}$ to a node of the graph, connecting them according to the distance between corresponding pixels [2]. The weight w_{ij} assigned to each edges is derived from the proposed inner product metric:

$$w_{ij} = \frac{1}{1+\eta h_{ij}^2},\ h_{ij} = \max\left\{\frac{\partial\mathcal{C}(P_i)}{\partial\overrightarrow{d_{ij}}}, \frac{\partial\mathcal{C}(P_j)}{\partial\overrightarrow{d_{ji}}}, 0\right\}, \frac{\partial\mathcal{C}(x)}{\partial\overrightarrow{d_{ij}}} = \langle\nabla\mathcal{C}(x), \overrightarrow{d_{ij}}\rangle,\ \overrightarrow{d_{ij}} = \frac{\overrightarrow{P_iP_j}}{|\overrightarrow{P_iP_j}|}, \quad (1)$$

where $\eta > 0$ is a tunning constant. The directional derivative of $\mathcal{C}$ in Eq. (1) considers the variation of the image in the directions defined by the edges of G, that is, $\overrightarrow{d_{ij}}$ and $\overrightarrow{d_{ji}}$.

Spectral Cut Partition. The spectral decomposition is carried out in this stage. More specifically, given the affinity graph G built from $\mathcal{C}$, we first decompose the graph Laplacian matrix as $\mathbf{L} = \mathbf{D} - \mathbf{W}$, where $\mathbf{D}$ and $\mathbf{W}$ contain the diagonal and off-diagonal elements of $\mathbf{L}$. Then, the Fiedler vector $\mathbf{f}$ is obtained by solving the generalized eigenvalue problem

$$(\mathbf{D} - \mathbf{W})\mathbf{x} = \lambda\mathbf{D}\mathbf{x}, \quad (2)$$

getting $\mathbf{f}$ as the eigenvector associated to the smallest non-zero eigenvalue. The Fiedler vector splits $\mathcal{C}$ into two subsets, one containing the pixels corresponding to nodes of the graph where the entries of $\mathbf{f}$ are positive and other containing the pixels with negative values of $\mathbf{f}$. For multiple partitions, the process is recursively performed until reaching the pre-defined number of clusters. Figure 1 portrays the spectral cut procedure.

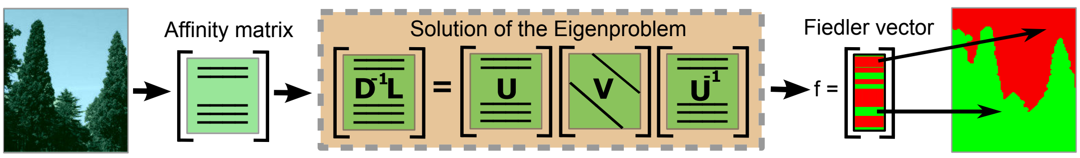

Figure 1: Spectral cut pipeline to partition the image from the zero-set of $\mathbf{f}$.

User Interactivity. The user can interactively change the partition initially obtained by stroking the resulting segmentation (see Figure 2). This step is performed by combining the texture component $\mathcal{T}$ initially generated in the Cartoon-Texture decomposition stage with a recent technique of harmonic analysis [24] aiming at incorporating the remaining high-level oscillatory information into the spectral cut process. See [3] for details.

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2.2 Experimental Results and Evaluations

We evaluate the performance of the proposed methodology by means of quantitative and visual analysis against two spectral segmentation state-of-the-art methods: *k-way Normalized Cuts method* (NCut) [6], and *Multiscale Normalized Cuts method* (MS-NCut) [19]. To accomplish the numerical evaluation, we make use of *Success Rate (Precision)* and *F-Score* measures [3] on the well-known *Berkeley Segmentation Dataset (BSDS)*, which provides 300 natural images with their human-drawn ground-truth segmentations. For a few illustrations and the measurements accomplished on BSDS dataset, see Figure 2.

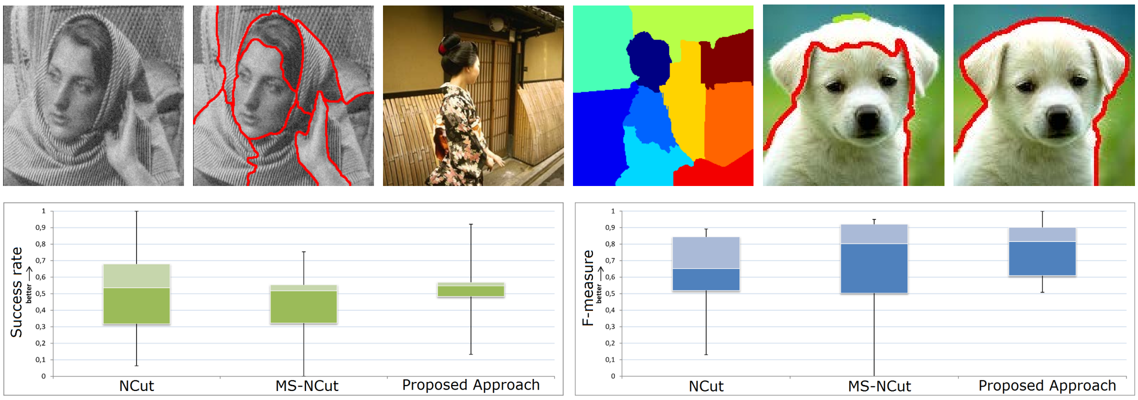

Figure 2: 1st row: Segmentation of a noise-textured image, example of multiple partition, and use of user interaction to improve the result. 2nd row: Quantitative comparison against NCut [6] and MS-NCut [19].

3 Laplacian Coordinates Energy Minimization on Graphs

We also propose a novel energy minimization clustering technique, first reported in [3, 5], that addresses many of the undesirable traits present in state-of-the-art methods such as non-uniqueness of solution to the associated optimization problem, use of computationally costly tools and the absence of an accurate and well-behaved (smoother) solution. The new approach, called *Laplacian Coordinates (LC)*, guarantees uniqueness of solution for the segmentation problem, presenting anisotropic behavior to ensure contour adherence on image boundaries. Moreover, the method allows for user intervention while leading to smoother and accurate solutions. Other important characteristic of Laplacian Coordinates is that the minimizer of the cost function is given by the solution of a constrained system of linear equations, making the algorithm quite simple to be used and coded [3].

3.1 Energy Minimization for Graph Clustering on Images

As a basic tool to compute the Laplacian Coordinates energy, we define a weighted graph $G = (V, E, W_E)$ from the image where V is the set of nodes corresponding to the image pixels, E is the edge set built from pairs of pixels locally connected in an 8-connected stencil, and W_E defines the set of weights. The LC energy E is then computed as follows:

$$E(\mathbf{x}) = \sum_{i\in B} \|x_i - x_B\|_2^2 + \sum_{i\in F} \|x_i - x_F\|_2^2 + \frac{1}{2}\sum_{(i,j)\in E} \|w_{ij}(x_i - x_j)\|_2^2, \tag{3}$$

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where $\mathbf{x} = (x_1, x_2, ..., x_n)$ is the saliency map which assigns a scalar value x_i to each pixel p_i of the image. The rationale behind the LC approach is that the non-pairwise terms from Equation (3) enforce fidelity of user-labeled pixels $p_i, i \in B \cup F$, to the scalars x_B (*background*) and x_F (*foreground*), respectively (see Fig.3 (middle)), while the last quadratic term imposes spatial smoothness within image segments and allows sharp jumps across image boundaries. Energy (3) is efficiently minimized by solving a sparse system of linear equations [3]. For a detailed discussion regarding Eq. (3), see the seminal works [3,5].

3.2 Experimental Results and Evaluations

In this section we provide a comparative evaluation of Laplacian Coordinates against five competing state-of-the-art methods, more specifically: *Graph Cuts (GC)* [25], *Power Watershed (PWS)*, *Maximum Spanning Forest (Kruskal's and Prim's algorithms - MSFK and MSFP)* [26], and *Random Walker (RW)* [4]. To quantitatively evaluate the results, we employ a set of well-established metrics on the classical benchmark *Grabcut* from Microsoft. As depicted in Fig.3 (bottom), the proposed approach quantitatively outperforms other competing techniques. A more comprehensive set of evaluations can be found in [3].

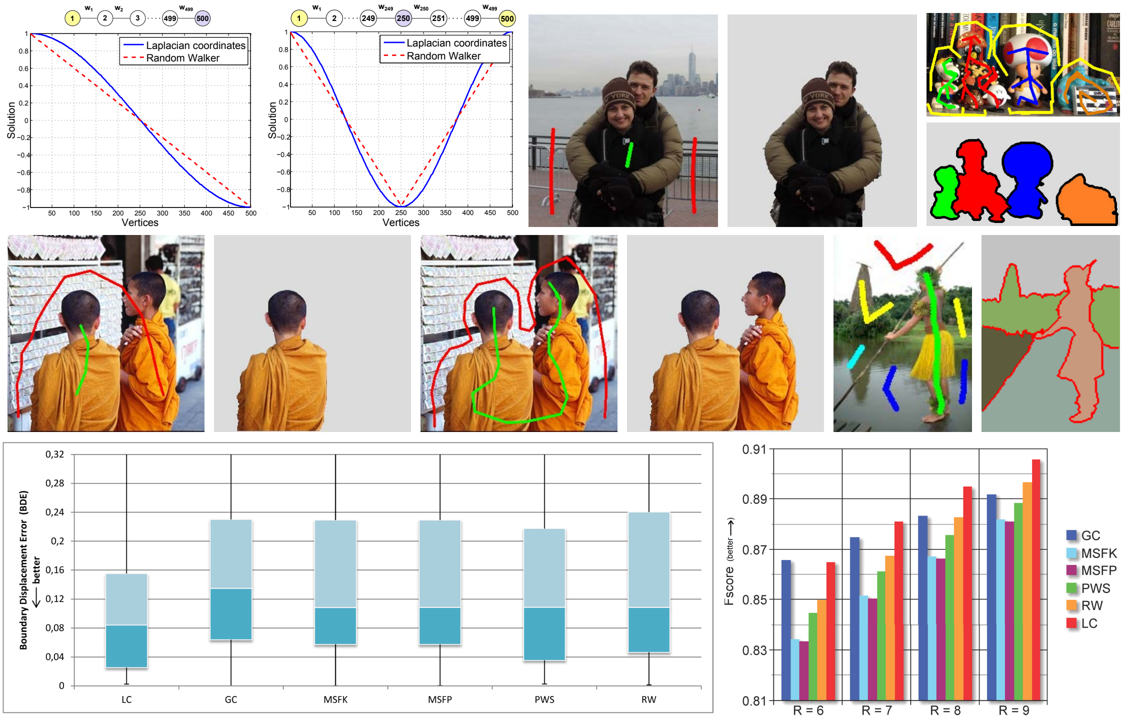

Figure 3: 1st row: (left) LC against RW on two unitary weighted graphs and (right/2nd row) results produced by our approach. 3rd row: comparison of our method (LC) against five state-of-the-art methods.

4 Image Inpainting and Photo Colorization

We also proposed in the thesis new algorithms for image inpainting and colorization. The methods rely on the accuracy of our segmentation apparatus and numerical PDEs to properly work, as reported in [3,8,14]. For general illustrations, see Fig. 4 and the videos[1].

[1](a) Inpainting examples: https://icmc.usp.br/e/4aac2 (b) Segmentation examples: http://icmc.usp.br/e/cd848

Figure 4: Inpainting and colorization illustrations obtained from the proposed methodologies [3,8,13,14].

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