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Indices of F-free graphs

Lilian Cavalet[1]
Instituto de Matemática e Estatística, UFRGS, Porto Alegre, RS
Luiz Emilio Allem[2]
Instituto de Matemática e Estatística, UFRGS, Porto Alegre, RS
Carlos Hoppen[3]
Instituto de Matemática e Estatística, UFRGS, Porto Alegre, RS

1 Introduction

In our work we study results about extremal graph theory that use a spectral view, in which properties of graphs are studied by means of eigenvalues of various matrices. We survey some recent applications of spectral graph theory tools to the classical Zarankiewicz Problem [4], we analyze the applicability of those methods to other matrices, and we raise questions for future research.

2 The Problem

Given a family $\mathcal{F}$ of graphs and an integer n, a Turán-type problem seeks to maximize the number of edges $e(G)$ of an n-vertex graph G that does not contain any subgraph $F \in \mathcal{F}$, that is, G is an F-**free graph**. In this work, we will consider only bipartite graphs that are free from complete bipartite subgraphs. This is known as the Zarankiewicz Problem.

Problem 1 (Zarankiewicz Problem)**.** *Let $z(m, n, s, t)$ be the maximum number of edges in a bipartite graph G with bipartition $X \cup Y$ where $|X| = m$ and $|Y| = n$, such that G does not contain a copy of the complete bipatite graph $K_{s,t}$ whose bipartition $S \cup T$ satisfies $S \subset X$ of size s and $T \subset Y$ of size t.*

3 Upper Bounds on Matrix Indices

In this work, we consider the following matrices associated with a graph G with vertex set $\{v_1, \ldots, v_n\}$. The **diagonal matrix** $D(G)$ is such that d_{ii} is the degree of v_i, and the **adjacency matrix** $A(G)$ is the $(0, 1)$-matrix such $a_{ij} = 1$ if and only if v_i and

[1]lilian.cavalet@ufrgs.br
[2]emilio.allem@ufrgs.br
[3]choppen@ufrgs.br

v_j are adjacent. Using these two matrices, we define the **Laplacian matrix** $L(G) = D(G) - A(G)$, and the **signless Laplacian matrix** $Q(G) = D(G) + A(G)$. If M is a matrix given by a graph G, the **index** $\lambda(M)$ is the largest eigenvalue of M.

In the table below Δ denotes the maximum degree of a graph, and $\frac{2e(G)}{n}$ is the average degree of the graph G, which will be used to establish a direct connection between the number of edges and the index. The conditions refer to $K_{s,t}$-free graphs, with $s \geq 2$.

	Condition	**Bound: index**	**Bound: edge**
A	$t = 2$	$\lambda(A) \leq \frac{1}{2} + \sqrt{(s-1)(n-1) + \frac{1}{4}}$ (1)	$\frac{2e(G)}{n} \leq \lambda(A)$
	$t \geq 3$	$\lambda(A) \leq \frac{1}{2}\frac{n(s-t+1)^{1/t}}{n^{1/t}} + \frac{(t-1)n}{n^{2/t}} + t - 2$ (2)	
L	$t = 1$	$\lambda(L) \leq \left(2 - \frac{1}{2s-2} + o(1)\right)\Delta$ (3)	$\frac{2e(G)}{n} \leq 3\Delta - \lambda(L)$
Q	$t = 2$	$\lambda(Q) \leq \frac{n+2(s-1)}{2} + \frac{1}{2}\sqrt{(n - 2(s-1)^2 + 8(s-1)}$ (4)	$\frac{2e(G)}{n} \leq \frac{\lambda(Q)}{2}$

Table 1: Bounds on the index of $K_{s,t}$-free graph matrices and the relation between each index and the number of edges.

Using spectral graph theory and linear algebra tools we studied in detail the four bounds presented above. We should mention that the bounds from the adjacency matrix give the best-known bounds for the Zarankiewicz Problem. The other bounds did not lead to improvements in the solution of **Problem 1**, but studying the index of families of graphs under some restrictions is an interesting problem on its own.

4 Future Work

After studying the main bounds for $K_{s,t}$-free graphs in the literature, we should verify if the necessary conditions imposed to the graphs could be relaxed and find, if possible, tighter bounds for the Laplacian matrix and signless Laplacian matrix. We are currently trying to extend the results for the normalized Laplacian matrix.

Also, we intend to study a modification of **Problem 1**, by replacing the forbidden graph $K_{s,t}$, by another graph, for example, a 1-regular graph.

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