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A conservative Lagrangian-Eulerian finite volume approximation method for balance law problems

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Abstract

We present a simple numerical method based on a Lagrangian-Eulerian framework for approximate solutions of nonlinear balance law problems. This framework has been used for numerically solving partial differential equations of several types, such as hyperbolic conservation laws [3, 8], balance laws problems [4]. As in [3, 5] the mass conservation takes place in the space-time volume D_j^n, and this region in the form of [3] is used to define naturally a balance law. This balance law is the central idea to build a efficient numerical method to approximate solution to balance law problems. Verification of the technique is also made by comparison with analytical solutions when they are available.

Keywords. hyperbolic balance laws, Lagrangian-Eulerian, Finite Volume Methods

1 Introduction

We propose a first order high-resolution three point numerical scheme based on a Lagrangian-Eulerian framework for numerically solving nonlinear balance law problems. The Lagrangian-Eulerian approach is a promising tool for numerically solving partial differential equations of several types. Recently, in [2,3,8] such ideas were extended to a wide range of nonlinear purely hyperbolic conservation laws and balance laws scalar and systems. Here, we make use of polynomial reconstruction ideas into the Lagrangian-Eulerian novel approach, but keeping the scheme simple, fast and without any strong restriction over the source term other than integrability on the finite volume. As in [2,3,8] the hyperbolic part of balance law is written in a space time divergence form so that the inherent conservation properties of the hyperbolic operator are used efficiently to build a numerical method to hyperbolic balance law problems [5]. Such framework presents an interesting property of being rather independent of a particular structure of the source terms. Our

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goal on the current work is two-fold: 1) to present the novel high-resolution Lagrangian-Eulerian three point numerical scheme for general balance laws, and 2) the application of the new scheme to a wide range of nonlinear balance laws that appear in transport in porous media problems as well as to the shallow water equations with discontinuous source term. For such problems, we present evidences that we are calculating qualitatively correct approximations with accurate resolution of small perturbations around the stationary solution. Our work shows accurate results computed efficiently with the simple high-resolution Lagrangian-Eulerian numerical scheme for general balance laws.

2 Numerical Method

We consider a novel Lagrangian-Eulerian formulation that can be viewed as an extension of previous works [2,3,8] for practical construction of numerical solutions for balance law problems, but following innovative recent ideas introduced in [1] to construct weak asymptotic methods for scalar equations and systems of conservation law equations. For simplicity, we consider the particular scalar equation with $u = u(x,t)$

$$\frac{\partial u}{\partial t} + \frac{\partial (u\, f(u))}{\partial x} = g(x,u), \quad x \in \mathbb{R},\, t > 0; \qquad u(x,0) = u_0(x) \in L^\infty(\mathbb{T}), \quad x \in \mathbb{R}, \tag{1}$$

with f Lipschitz, with Lipschitz coefficient bounded on bounded sets and source term $g(x,u)$ integrable over the finite volume D_j^n. We provide a formal development of the analogue Lagrangian-Eulerian scheme [4,5,8] for numerically solving the initial value problem (1). As in the Lagrangian-Eulerian schemes [3–5], a local mass balance equation is obtained by integrating the hyperbolic balance law (1) over the region in the space-time domain. Here we consider the Lagrangian-Eulerian finite-volume cell centers

$$D_j^n = \{(t,x) \ / \ t^n \leq t \leq t^{n+1},\ \sigma_{j-\frac{1}{2}}(t) \leq x \leq \sigma_{j+\frac{1}{2}}(t)\}, \tag{2}$$

where $\sigma^n_{j-\frac{1}{2}}(t)$ is the parameterized integral curve such that $\sigma^n_{j-\frac{1}{2}}(t^n) = x^n_{j-\frac{1}{2}}$. These curves are the lateral boundaries of the domain D_j^n in (2) and we define $\bar{x}^{n+1}_{j-\frac{1}{2}} := \sigma^n_{j-\frac{1}{2}}(t^{n+1})$ and $\bar{x}^{n+1}_{j+\frac{1}{2}} := \sigma^n_{j+\frac{1}{2}}(t^{n+1})$ as their endpoints in time t^{n+1}. The numerical scheme is expected to satisfy some type of local mass balance (due to the inherent nature of the problem) from time t^n in the space domain $\left[x^n_{j-\frac{1}{2}}, x^n_{j+\frac{1}{2}}\right]$ to time t^{n+1} in the space domain $\left[\bar{x}^{n+1}_{j-\frac{1}{2}}, \bar{x}^{n+1}_{j+\frac{1}{2}}\right]$. With this, we must have the flux through curves $\sigma^n_{j-\frac{1}{2}}(t)$ to be zero. From the integration of (1) and the divergence theorem applied on the hyperbolic operator, left side of equation (1), and using the fact that the line integrals over curves $\sigma^n_j(t)$ vanish, we get the local balance mass equation

$$\int_{\bar{x}^{n+1}_{j-\frac{1}{2}}}^{\bar{x}^{n+1}_{j+\frac{1}{2}}} u(x,t^{n+1})dx = \int_{x^n_{j-\frac{1}{2}}}^{x^n_{j+\frac{1}{2}}} u(x,t^n)dx + \iint_{D_j^n} g(x,u)\, dxdt. \tag{3}$$

As in [3], the curves $\sigma^n_{j-1/2}(t)$ are not straight lines in general, but rather solutions of the local nonlinear differential equations [2,4,8]: $\frac{d\sigma^n_{j-1/2}(t)}{dt} = \frac{u\,f(u)}{u} = f(u)$, for $t^n < t \leq t^{n+1}$, with initial condition $\sigma^n_{j-1/2}(t^n) = x^n_{j-1/2}$, assuming $u \neq 0$ (for the sake of presentation).

The extension of this construction follows naturally from the finite volume formulation of the linear Lagrangian-Eulerian scheme, as in [2,3,8], building block to construct *local* approximations such as $f^n_{j-1/2} = f(U^n_{j-1/2}) \approx f(u(x_{j-\frac{1}{2}}, t^n))$ with the initial condition $\sigma^n_{j-1/2}(t^n) = x^n_{j-1/2}$. Indeed, distinct and high-order approximations are also acceptable for $\frac{d\sigma^n_{j-1/2}(t)}{dt} = f(u)$. As in [3], the piecewise constant numerical data is reconstructed into a piecewise linear approximation (but high-order reconstructions are acceptable), through the use of MUSCL-type interpolants $L_j(x,t) = u_j(t) + (x - x_j)\frac{1}{\Delta x}u'_j$. For the numerical derivative $\frac{1}{\Delta x}u'_j$, there are several choices of slope limiters. A priori choice of such slope limiters is quite hard, but they are chosen upon the underlying model problem under investigation. A possible slope limiter is

$$U'_j = MM\left\{\alpha\Delta u_{j+\frac{1}{2}}, \frac{1}{2}(u_{j+1} - u_{j-1}), \alpha\Delta u_{j-\frac{1}{2}}\right\}, \tag{4}$$

and this choice allows steeper slopes near discontinuities and retain accuracy in smooth regions. The range of the parameter α is typically guided by the CFL condition. Equation (3) defines a local mass balance between space intervals at time t^n and at time t^{n+1}. We will later address how to project these volumes back to the original mesh. Defining

$$\overline{U}^{n+1}_j := \frac{1}{h^{n+1}_j}\int_{\bar{x}^{n+1}_{j-\frac{1}{2}}}^{\bar{x}^{n+1}_{j+\frac{1}{2}}} u(x,t^{n+1})dx, \quad \text{and} \quad U^n_j := \frac{1}{h}\int_{x^n_{j-\frac{1}{2}}}^{x^n_{j+\frac{1}{2}}} u(x,t^n)dx,$$

then, equation (3) can be rewritten into

$$\overline{U}^{n+1}_j \;\; = \frac{1}{h^{n+1}_j}\left(h\,U^n_j + \iint_{D^n_j} g(x,u)\,dxdt\right). \tag{5}$$

Solutions $\sigma^n_{j-1/2}(t)$ of the differential system are obtained using the approximations

$$U_{j-\frac{1}{2}} \;\; = \tfrac{1}{h}\left(\int_{x^n_{j-1}}^{x^n_{j-\frac{1}{2}}} L_{j-1}(x,t)dx + \int_{x^n_{j-\frac{1}{2}}}^{x^n_j} L_j(x,t)dx\right) = \frac{1}{2}(U_{j-1} + U_j) + \frac{1}{8}(U'_j - U'_{j-1}). \tag{6}$$

The above approximation is not necessary in the linear case where $u\,f(u) = a(x,t)\,u$. We must notice that the approximation of $f^n_{j-1/2}$ may cause spurious oscillation in Riemann problems, specially in shocks and discontinuity regions. For that, we use a polynomial reconstruction of second degree to smooth out the approximation and also slope limiters approximation of the form (4). The numerical solutions have shown qualitatively correct behavior for nonlinear hyperbolic conservation laws. Convergence order remains unchanged even with the reconstruction, being first-order. In the reconstruction we use the nonlinear Lagrange polynomial in U_{j-1}, U_j and U_{j+1}. Equation (5) reads,

$$\overline{U}^{n+1}_j = \frac{1}{h^{n+1}_j}\int_{x^n_{j-\frac{1}{2}}}^{x^n_{j+\frac{1}{2}}} P_2(x)dx, \tag{7}$$

where $P_2(x) = U_{j-1}^n\, L_{-1}(x-x_j) + U_j^n\, L_0(x-x_j) + U_{j+1}^n\, L_1(x-x_j)$, $L_0(x) = 1 - \left(\frac{x}{h}\right)^2$ and $L_{\pm 1}(x) = \frac{1}{2}\left[\left(\frac{x}{h} \pm \frac{1}{2}\right)^2 - \frac{1}{4}\right]$. Next, we obtain the resulting projection formula as follows

$$U_j^{n+1} = \overline{U}_j^{n+1} + \frac{\Delta t^n}{h}\left(f_{j-\frac{1}{2}}^{+}\overline{U}_{j-1}^{n+1} - |f_j|\overline{U}_j^{n+1} + f_{j+\frac{1}{2}}^{-}\overline{U}_{j+1}^{n+1}\right), \tag{8}$$

where $f_{j-\frac{1}{2}}^{+} = f^{+}(U_{j-\frac{1}{2}}^n)$, $f_{j+\frac{1}{2}}^{-} = f^{-}(U_{j+\frac{1}{2}}^n)$ and $|f_j| = f_{j-\frac{1}{2}}^{+} + f_{j+\frac{1}{2}}^{-}$. Here Δt^n is obtained under CFL-condition

$$\max_j \left\{ f_{j-\frac{1}{2}}^{+}, f_{j+\frac{1}{2}}^{-} \right\} \Delta t^n \leq \frac{h}{2},$$

which is taken by construction of method. We note that in the linear case, when $a(x,t) = a > 0$ (or $a < 0$), the numerical scheme (5)-(8) is a extension of the Upwind scheme to linear balance law, but our scheme can approximate solution in both cases $a > 0$ and $a < 0$, the CFL-condition in this case is $|a\,\Delta t| \leq h$ as in the Upwind scheme.

3 Numerical Experiments

We present and discuss approximate computations for scalar balance laws and systems of balance laws. The scalar calculations were performed in less that one second with Matlab on a standard laptop with 2.60 GHz Intel Core i7-4510U CPU and 8.0 GB of RAM memory.

The first test is an example of linear advection with a smooth (polynomial) source:

$$u_t + 2u_x = x^3 + 6tx^2, \qquad u(x,0) = 0.$$

The initial data here is zero, but the exact solution of this differential equation is $u(x,t) = tx^3$. In $x = 0$ we have a sonic point accurately captured by our simulations. Figure 1 presents numerical solutions at times $t = 0$, $t = 1.5$ and $t = 3.0$ for a 256 cells mesh. For this case we have a natural and robust generalization for the upwind method for balance laws. The observed convergence rate was studied at time $t = 3.0$ with $32, 64, 128, 256, 512$ and 1024 mesh grid cells and second-order convergence was observed (see Figure 2). Here we used the midpoint rule for the source term quadrature, but the linear advection hyperbolic operator is being exactly calculated due to exact CFL condition.

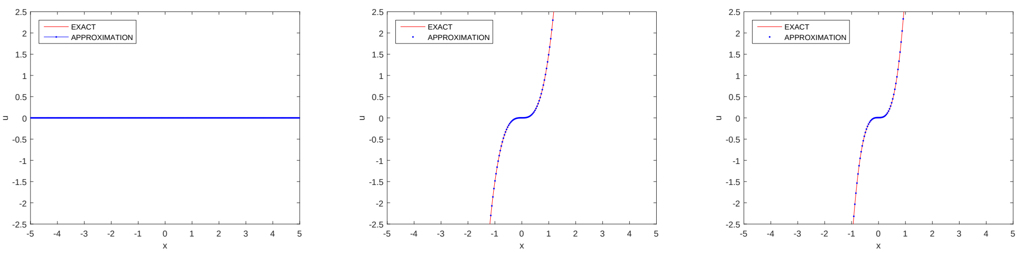

Figure 1: Numerical solutions with smooth source term $g(x,t) = x^3 + 6tx^2$.

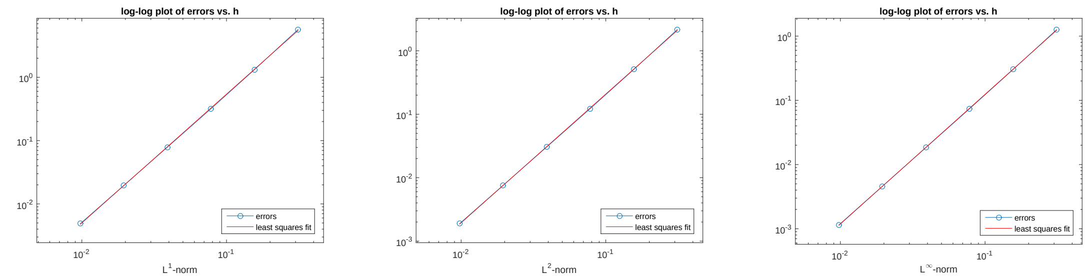

Figure 2: Convergence of error in L^1, L^2 and L^∞ norms with uniform mesh refinement for the smooth source term test. Second-order convergence is observed in this example.

For the second test, proposed by Shi Jin in [9], the source term is of the discontinuous form $g(x,u) = z'(x)u$.

$$u_t + (u\,f(u))_x = g(x,u)$$

with flux function $uf(u) = \frac{u^2}{2}$ and $z(x) = \cos(\pi\, x)$, $4.5 \le x \le 5.5$ and 0 otherwise with $0 < x < 10$. Note that $z'(x)$ is a discontinuous function, so that $g(x,u)$ is a discontinuous source term in x. Figure 3 involves approximations with initial data $u(x,0) = 0$, $x > 0$ and $u(0,t) = 2$, $t > 0$. The steady state solution of this problem is $u + z = 2$. The pictures in Figure 3 show approximations with 128 cells (left), 256 (middle) and 512 cells (right) for u (top pictures) and for steady state $u + z$ (bottom pictures). The numerical results present clearly qualitatively correct approximations at $t = 1$.

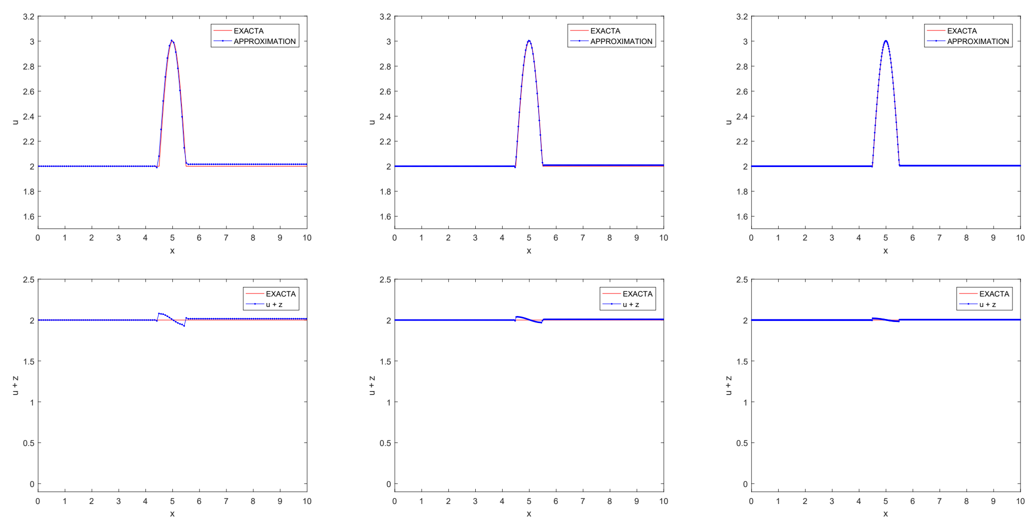

Figure 3: Numerical solutions with discontinuous source term $g(x,u)$.

We also consider, as in [7], a 2×2 nonlinear system of balance laws modeling the flow of water downing in a channel having a rectangular cross section and inclined at a constant angle θ to the horizontal. This is a prototype model for shallow-water flow (see [8]) in an

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inclined channel with friction the system may be written as (in dimensionless variables)

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial (hu)}{\partial x} = 0, \\[2ex] \dfrac{\partial (hu)}{\partial t} + \dfrac{\partial \left(hu^2 + \frac{1}{2}h^2\right)}{\partial x} = h - C\dfrac{1+h}{\tan(\theta)}v^2, \end{cases} \tag{9}$$

where h is the height of the free surface and v is the averaged horizontal velocity. Precisely, as in [7], the friction coefficient C is taken to be 0.1, while the inclination angle $\theta = \frac{\pi}{6}$. On physical grounds, in this model problem it was assumed the hydrostatic balance in the vertical direction and ignored any surface tension. Here, the initial velocity is taken to be $v_0 = 1.699$, while the initial height of the free surface consists of a triangular perturbation of the uniform flow level, $h_0(x) = x + 1.5$, $-0.5 \leq x \leq 0$, $h_0(X) = -x + 1.5$, $0 \leq x \leq 0.5$, and 1 elsewhere. Numerical approximations are shown in Figure 4 with a clearly qualitatively correct approximations at $t = 1$.

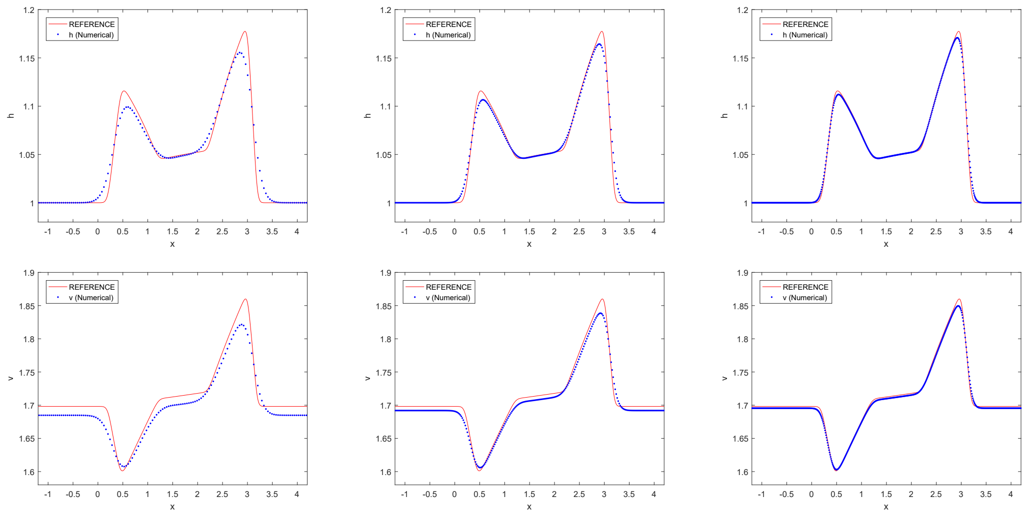

Figure 4: Numerical solutions to shallow water system (9) with 128, 256 and 512 cells (left to right), h (height) in top and v (velocity) bottom .

4 Concluding Remarks

We presented the development of a simple and effective numerical scheme for solving nonlinear scalar balance laws problems with the Lagrangian-Eulerian framework. This method is based on a reformulation of the conservation laws in terms of an equivalent locally conservative space-time problem in divergence form. We make use of piecewise linear and parabolic reconstructions ideas for resolution and accuracy reasons and the resulting method present qualitatively correct numerical approximations. Our method is robust in a way that no special treatment is needed when the sign of velocity changes over time. We expect to establish a componentwise extension of the scheme in order to perform numerical experiments for systems of conservation and balance laws, as well as multidimensional problems. Our numerical experiments show good evidence of computational convergence and preservation of the well-balanced property of balance laws.

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