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Fractional Differential Operators: Eigenfunctions

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Abstract. Eigenfunctions associated with Riemann-Liouville and Caputo fractional differential operators are obtained by imposing a restriction on the fractional derivative parameter.

Keywords. Riemann-Liouville fractional derivatives, Linear fractional differential equations, Caputo fractional derivative and Mittag-Leffler functions.

1 Introduction

The Non-Integer Order Calculus, traditionally known as Fractional Calculus (FC), is the branch of mathematics that deals with the study of integrals and derivatives of non-integer order. Althouhg it is not accurate, since the order of an integral and a derivative can be real and also complex [23, 24], it has played an outstanding role [24] since its creation, as several mathematicians and applied researchers have obtained important results by modeling real processes using FC [17, 19].

Given a differential equation that describes a specific phenomenon, a common way to use fractional modeling is to replace the integer order derivatives by non-integer derivatives, usually with order lower than or equal to the order of the original derivatives, so that the usual solutions may be recovered as a particular cases [7].

Although there is no trivial physical and geometrical interpretation for the fractional derivative and the fractional integral [22], fractional order differential equations are naturally related to systems with memory, as fractional derivatives are usually nonlocal operators, *i.e.* the calculation of a time-fractional derivative at a given time requires its knowledge at all previous times [7, 19]. Processes with memory exist in many biological systems [16]. Besides, fractional differential equations may help in reducing the errors arising from the neglected parameters in modeling real life phenomena [1, 14, 17].

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There are several applications of fractional calculus in engineering, for example in the study of control and dynamical systems [19]. Moreover, there are in physics several potential applications of fractional derivatives, for instance, in the generalization of classical equations [2–5].

In medicine, it has been found that the electrical conductance of the cell membranes of living organisms are described by fractional order equations, so that they may be classified in groups of non-integer order models. Fractional derivatives embody essential features of cell rheological behavior and have enjoyed a great success in the field of rheology [1]. Some mathematical models in HIV show that fractional models are more approximate than their integer order versions [1, 9].

With the aim of solving fractional partial differential equations and generalize results, several definitions of "fractional derivative"(FD) have been proposed [8]. For example, one should also consider the Grünwald-Letnikov, Riemann-Liouville and Caputo fractional derivatives and the Riesz potential. A natural question arises: "What is a fractional derivative?"In a paper whose title is exactly this question [18], Manuel D. Ortigueira and J. A. Tenreiro Machado set a criterion named *Wide Sense Criterion* (WSC) which establishes when an operator is a FD and showed that the well-known definitions of Grünwald-Letnikov, Riemann-Liouville, Caputo and Riesz satisfy the WSC.

Since linear sequential fractional differential equations are naturally related to the modeling of many problems in physics and applied science, several methods to obtain the analytical solutions for this kind of equations have been studied [21].

This work presents, with the help of some particular linear fractional differential equations, new theorems about eigenfunctions related to the fractional differential operators of Riemann-Liouville, $\mathrm{D}_{a+}^{n\alpha}$ and of Caputo ${}^{\mathrm{C}}\mathrm{D}_{a+}^{n\alpha}$. From these theorems, we may express the exponential function in terms of a sum involving the Mittag-Leffler functions[4] [12].

The results are presented as follows: after this introduction, In Section 2 the definition of the so-called linear sequential fractional differential equation and new theorems are introduced. In Section 3 we present the analytical solution of some particular linear sequential fractional differential equations. Finally, Section 4 brings the concluding remarks.

2 Linear fractional differential equations

In this paper we employ the left-sided fractional operators explained above. A similar reasoning will extend the results obtained here to right-sided fractional operators[5].

Let $\Lambda = [a, b]$ be an interval on the real axis $\mathbb{R}$, $x \in \Lambda$, $0 < \alpha \leq 1$, $n \in \mathbb{N}$ and g, $a_j : \Lambda \to \mathbb{R}$ continuous functions for $j = 0, 1, 2, \ldots, n-1$. The linear sequential fractional differential equations [15, 21] of order $n\alpha$, denoted by $[\mathrm{L}_{n\alpha}(f)](x)$, are defined as

[4]These result can be found at [13].

[5]For the basic aspects and definitions of FC see [7].

$$[\mathrm{L}_{n\alpha}(f)]\,(x) := \left(D^{n\alpha}_{a+}f\right)(x) + \sum_{j=0}^{n-1} a_j(x)\left(D^{j\alpha}_{a+}f\right)(x) = g(x), \tag{1}$$

where D^{α}_{a+} represents the fractional derivative of Riemann-Liouville (or Caputo) and

$$\left(D^{0}_{a+}f\right)(x) = f^{(0)}(x) = f(x).$$

In this paper we consider the particular case in which $a_0(x) = \lambda$, with $\lambda \neq 0$, $a_j(x) = 0$, for $j = 1, 2, \ldots, n-1$ and $g(x) = 0$ in equation (1). As a result, we obtain a homogeneous linear sequential fractional differential equation, denoted by $[\hat{\mathrm{L}}_{n\alpha}(f)](x)$ and defined as

$$[\hat{\mathrm{L}}_{n\alpha}(f)](x) := \left(D^{n\alpha}_{a+}f\right)(x) - \lambda f(x) = 0. \tag{2}$$

From the two theorems presented below about eigenfunctions of the Riemann-Liouville and Caputo fractional operators it is possible to find solutions for fractional differential equations of the type of equation (2), where D^{α}_{a+} can be the Riemann-Liouville or the Caputo fractional operators[6].

Theorem 2.1.. *Let $\Omega = (a, b)$ be an interval on the real axis $\mathbb{R}$, $n \in \mathbb{N}$, $\lambda \neq 0$ a real number and $\alpha \in \left(\frac{n-1}{n}, 1\right]$ an interval on the real axis. Then*

$$f_k(x) = (x-a)^{n\alpha-k}\, E_{n\alpha, n\alpha+1-k}\left[\lambda (x-a)^{n\alpha}\right], \tag{3}$$

for $x \in \Omega$ and $k = 1, 2, \ldots, n$, are the eigenfunctions associated with the eigenvalues λ of the Riemann-Liouville left-sided fractional derivative $D^{n\alpha}_{a+}$.

Now, considering the Caputo left-sided fractional derivative [13], we have the following theorem:

Theorem 2.2.. *Let $\Lambda = [a, b]$ be an interval on the real axis $\mathbb{R}$, $n \in \mathbb{N}$, $\lambda \neq 0$ a real number and $\alpha \in \left(\frac{n-1}{n}, 1\right]$ an interval on the real axis. Then*

$$\varphi_k(x) = (x-a)^{k-1}\, E_{n\alpha, k}\left[\lambda (x-a)^{n\alpha}\right], \tag{4}$$

with $x \in \Lambda$ and $k = 1, 2, \ldots, n$, are the eigenfunctions associated with the eigenvalues λ of the Caputo left-sided fractional derivative ${}^{C}D^{n\alpha}_{a+}$.

Proposition 2.1.. *If $\{y_k(x)\}_{k=1}^{n}$ is a fundamental system of solutions for the linear sequential fractional differential equation (1) with $g(x) = 0$, then the general solution of the equation is given by [15]*

$$y(x) = \sum_{k=1}^{n} c_k y_k(x), \tag{5}$$

where $\{c_k\}_{k=1}^{n}$ are arbitrary constants.

[6]The proof the following theorems, corollaries and propositions can be found at [13]

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Corollary 2.1. *Let $\Omega = (a, b)$ be an interval on the real axis $\mathbb{R}$, n a natural number and $\alpha \in \mathbb{R}$ $\left(\frac{n-1}{n} < \alpha \leq 1\right)$. Then the general solution of the linear sequential fractional differential equation*

$$\left(D_{a+}^{n\alpha} f\right)(x) - \lambda f(x) = 0 \tag{6}$$

can be written as

$$f_\alpha(x) = \sum_{k=1}^{n} c_k (x-a)^{n\alpha-k} E_{n\alpha, n\alpha+1-k}\left[\lambda (x-a)^{n\alpha}\right], \tag{7}$$

where $\{c_k\}_{k=1}^{n}$ are arbitrary constants and $x \in \Omega$.

Corollary 2.2.. *Let $\Lambda = [a, b]$ be an interval on the real axis $\mathbb{R}$, n a natural number and $\alpha \in \mathbb{R}$, $\left(\frac{n-1}{n} < \alpha \leq 1\right)$. Then the general solution of the linear sequential differential equation*

$$\left({}^{C}D_{a+}^{n\alpha} \varphi\right)(x) - \lambda \varphi(x) = 0 \tag{8}$$

can be written as

$$\varphi_\alpha(x) = \sum_{k=1}^{n} c_k (x-a)^{k-1} E_{n\alpha, k}\left[\lambda (x-a)^{n\alpha}\right], \tag{9}$$

where $\{c_k\}_{k=1}^{n}$ are arbitrary constants and $x \in \Lambda$.

3 Differential equation $[\hat{\mathbf{L}}_{n\alpha}(f)](x)$

There are recent applications of FC in different fields of knowledge and several methods have been proposed for solving fractional differential equations [20].

In this section we present some examples of equations arising from equations (6) and (8).

Example Let $\Omega = (0, b)$ be an interval on the real axis $\mathbb{R}$ and $\alpha \in \mathbb{R}$ $\left(\frac{2}{3} < \alpha \leq 1\right)$. Using Corollary 2.1, the solution of the fractional differential equation

$$\left(\mathrm{D}_{0+}^{3\alpha} f\right)(x) - f(x) = 0, \tag{10}$$

can be written as

$$f_\alpha(x) = c_1\, x^{3\alpha-1} \mathrm{E}_{3\alpha,3\alpha}\left(x^{3\alpha}\right) + c_2\, x^{3\alpha-2} \mathrm{E}_{3\alpha,3\alpha-1}\left(x^{3\alpha}\right) + c_3\, x^{3\alpha-3} \mathrm{E}_{3\alpha,3\alpha-2}\left(x^{3\alpha}\right), \tag{11}$$

where $\{c_k\}_{k=1}^{3}$ are arbitrary constants and $x \in \Omega$.

Example Let $\Lambda = [0, b]$ be an interval on the real axis $\mathbb{R}$, $\alpha \in \mathbb{R}$ $\left(\frac{1}{2} < \alpha \leq 1\right)$, and $\lambda \neq 0$ a real constant. From Corollary 2.2, the solution of the following homogeneous fractional Helmholtz equation in one variable, which represents a time-independent form of the wave equation,

$$\left(\mathrm{D}_{0+}^{2\alpha} \varphi\right)(x) + \lambda^2 \varphi(x) = 0, \tag{12}$$

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can be written as

$$\varphi_\alpha\left(x\right) = c_1\, x^{2\alpha-1}\mathrm{E}_{2\alpha,2\alpha}\left(-\lambda^2 x^{2\alpha}\right) + c_2\, x^{2\alpha-2}\mathrm{E}_{2\alpha,2\alpha-1}\left(-\lambda^2 x^{2\alpha}\right), \tag{13}$$

where c_1 and c_2 are arbitrary constants.

From [12], we know that

$$\mathrm{E}_2\left(z\right) = \cosh\left(\sqrt{z}\right) \qquad \text{and} \qquad \mathrm{E}_{2,2}\left(z\right) = \frac{\sinh\sqrt{z}}{\sqrt{z}}.$$

Thus

$$\mathrm{E}_2\left(-\lambda^2 x^2\right) = \cosh\left(i\ |\lambda|\, x\right) = \cos\left(|\lambda|\, x\right), \tag{14}$$

and

$$\mathrm{E}_{2,2}\left(-\lambda^2 x^2\right) = \frac{\sinh\left(i\ |\lambda|\, x\right)}{i\ |\lambda|\, x} = \frac{\sin\left(|\lambda|\, x\right)}{|\lambda|\, x}. \tag{15}$$

If $\alpha = 1$ in equation (12), using equations (14) and (15) into equation (13), we obtain a particular solution for the Helmholtz equation,

$$\Delta\varphi + \lambda^2\varphi = 0, \tag{16}$$

namely,

$$\varphi_1\left(x\right) = c_1\, \cos\left(|\lambda|\, x\right) + c_2\, \frac{\sin\left(|\lambda|\, x\right)}{|\lambda|}, \tag{17}$$

where $\{c_k\}_{k=1}^{2}$ are arbitrary constants and $x \in \Lambda$.

The graphic of the solution given in equation (13) is presented for different values of α $\left(\frac{1}{2} < \alpha \leq 1\right)$ assuming $c_1 = c_2 = 1$ and $\lambda = 2$.

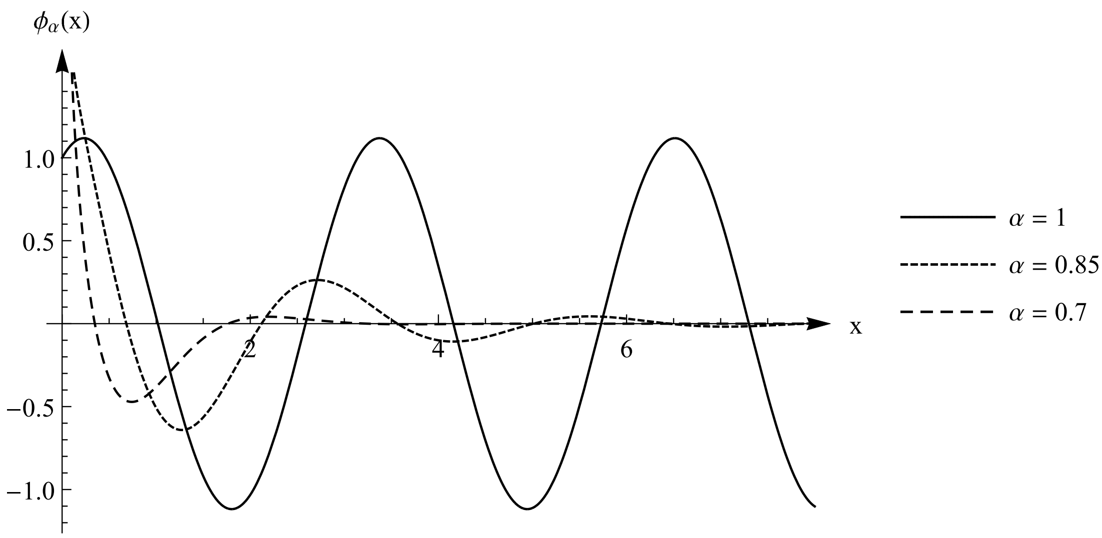

Figura 1: Curves for $\varphi_\alpha\left(x\right)$, for different values of α.

4 Concluding remarks

This work presents the so-called eigenfunctions associated with the Riemann-Liouville and Caputo fractional differential operators, considering the parameter $n\alpha$ with n a natural number and α in a limited interval that depends on n. In both cases the eigenfunctions were obtained in terms of the two parameter Mittag-Leffler functions. Besides that, the analytical solution of a particular homogeneous linear sequential fractional differential equations of the kind $[\hat{\mathrm{L}}_{n\alpha}(f)](x)$ was obtained through the eigenfunctions found in Theorems 2.1 and 2.2. Examples and applications were discussed.

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