Proceeding Series of the Brazilian Society of Applied and Computational Mathematics, v. 6, n. 1, 2018.

Trabalho apresentado no XXXVII CNMAC, S.J. dos Campos - SP, 2017.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

On the problem of mixture of elastic materials with frictional damping

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Abstract. We consider a mathematical model that describes a mixture of n elastic materials with frictional damping and boundary frictional damping. We study, from the functional and numerical analysis point of view, the mixture problem with frictional damping. The problem consists of a linear system of n coupled hyperbolic partial differential equations. An existence and uniqueness result and an energy decay property are mentioned. In the case of boundary frictional damping we show that the corresponding semigroup is exponentially stable if $\mathbf{B}$ (dissipation parameters) is full rank. Then, a fully discrete approximation is introduced using the finite-difference method to characterize system energy.

Keywords. Exponential Stability, Semigroups, Mixture of Materials, Finite Difference Method, Numerical Simulations.

1 Introduction

The theory of mixtures of solids has been widely investigated in the last decades [1,2,5,6]. In recent years, an increasing interest has been directed to the study of the qualitative properties of solutions related to mixtures composed of two interacting continua. Several results concerning existence, uniqueness, initial-condition continuous dependence and asymptotic stability can be found in the literature [9,10,12]. In [4], it was made a full characterization of the asymptotic behavior of the following mixture model

$$\mathbf{R}U_{tt} - \mathbf{A}U_{xx} + \mathbf{B}U_t = 0, \tag{1}$$

with $U = (u^1, \cdots, u^n)$, $\mathbf{R} = (\rho_i\delta_{ij})_{n\times n}$, $\quad \mathbf{A} = (a_{ij})_{n\times n}$, $\quad \mathbf{B} = (b_{ij})_{n\times n}$. Where δ_{ij} is the Kronecker's delta, $\mathbf{A}$ is a positive definite (real) symmetric matrix and $\mathbf{B}$ a semipositive definite (real) symmetric matrix. They proved that depending on the relationship of the coefficients, two situations may occur when Dirichlet boundary conditions is considered. The system can be exponentially stable or there exists an oscillating solution.

Let $\mathcal{S}_{\mathcal{A}}(t)$ be the semigroup associated to (1) then

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Theorem 1.1. *Let us denote by $\mathcal{W} = \mathbf{R}^{-1}\mathbf{A}$ and let $\mathbf{B}$ be a positive semidefinite matrix, then the following statements are equivalents.*

- *$\mathcal{S}_{\mathcal{A}}(t)$ is exponentially stable.*
- *Denoting by $\mathbf{B}_j$ the j-row vector of $\mathbf{B}$ then*

$$dim\ span\left\{\mathbf{B}_j,\ \mathbf{B}_j\mathcal{W},\ \mathbf{B}_j\mathcal{W}^2,\ ...,\ \mathbf{B}_j\mathcal{W}^{n-1},\ j=1,2...,n\right\} = n. \tag{2}$$

Proof. See [4].

As an straightforward corolary, the dissipative mechanism is described by the rank of the matrix $\mathbf{B}$.

For now on, we study the one dimensional model of a mixture of n interacting continua over the compact interval $[0,l]$. We consider a localized frictional dissipation at $x = l$,

$$\mathbf{R}U_{tt} - \mathbf{A}U_{xx} = 0, \tag{3}$$

$$U(0,t) = 0 \quad \text{and} \quad \mathbf{A}U_x(l,t) + \mathbf{B}U_t(l,t) = 0, \quad t \in \mathbb{R}^+, \tag{4}$$

$$U(x,0) = U_0(x), \quad U_t(x,0) = U_1(x). \tag{5}$$

Without loss of generality we can assume that $\mathbf{R} = \mathbf{I}$ and $\mathbf{B}$ is diagonal matrix, otherwise we make the substitution $U = \mathbf{S}^T\widetilde{U}$ in equations (3)-(4), where $\mathbf{S}^T$, the transpose $\mathbf{S}$, is a non-singular matrix that diagonalize $\mathbf{R}$ and $\mathbf{B}$ simultaneously [3].

The questions that arrise are: is it possible that system (3)–(5) is exponentially stable? Or, there exists oscillating solutions? To answer theses questions, the rank of $\mathbf{B}$ play an important role, as shown later. The main result of this study is that the semigroup associated to (3)-(5) is exponentially stable if $\mathbf{B}$ is a positive definite.

2 Semigroup formulation

The semigroup theory is used to show the well posedness as well as the asymptotic properties of (3)–(5). We denoted $\mathcal{H}^1_\star(0,l) = \{U \in [H^1(0,l)]^n\ ,\ U(0) = 0\}$ and $\mathcal{L}^2(0,l) = [L^2(0,l)]^n$. To do that, let us introduce the phase space $\mathbb{H} = \mathcal{H}^1_\star(0,l) \times \mathcal{L}^2(0,l)$, which is a Hilbert space with the norm

$$||(U,V)||^2_{\mathbb{H}} = \int_0^l U_x^*\mathbf{A}U_x dx + \int_0^l V^*\mathbf{R}V dx \tag{6}$$

Let us introduce the operator $\mathcal{A}$ given by

$$\mathcal{A}\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} V \\ \mathbf{R}^{-1}\mathbf{A}U_{xx} \end{pmatrix} \tag{7}$$

with domain $D(\mathcal{A}) = \left\{(U,V) \in \mathcal{H}^1_\star(0,l) \cap \mathcal{H}^2(0,l) \times \mathcal{H}^1_\star(0,l)\ ;\ \mathbf{A}U_x(l) + \mathbf{B}V(l) = 0\right\}$. Under this conditions the initial-boundary value problem (3)-(5) can be rewritten as

$$\frac{d}{dt}\mathbf{U} = \mathcal{A}\mathbf{U}, \quad \mathbf{U}(0) = \mathbf{U}_0, \tag{8}$$

where $\mathbf{U}(t) = (U(t), V(t))^T$ and $\mathbf{U}_0 = (U_0, U_1)^T$.

The solubility of the abstract Cauchy problem (8) is guaranteed by the next Theorem, which can be demonstrated similarly to Theorem 4 of [4].

Theorem 2.1. *The operator $\mathcal{A}$ is the infinitesimal generator of a contractions C_0-semigroup, denoted by $\mathcal{S}_{\mathcal{A}}(t) = e^{\mathcal{A}t}$.*

Another important tool used for the characterization of the exponential stability of a C_0-semigroup was obtained by [7] and [11] independently. Here we use the version due to [11].

Theorem 2.2. *Let $\mathcal{S}_A(t)$ be a C_0-semigroup of contractions of linear operators on Hilbert space $\mathcal{H}$ with infinitesimal generator A. Then $\mathcal{S}_A(t)$ is exponentially stable if and only if*

$$i\mathbb{R} \subset \varrho(A) \quad \textit{and} \quad \limsup_{|\lambda|\to+\infty} ||(i\lambda I - A)^{-1}||_{\mathcal{L}(\mathcal{H})} < \infty$$

where $\mathcal{L}(\mathcal{H})$ denotes the space of continuous linear functions in $\mathcal{H}$.

Theorems 2.1 and 2.2 are the basic tools for the well posedness and the exponential stability of system (3)–(5).

3 On the Stability of Semigroup

The objective of this section is to demonstrate that the exponential stability is achieved when $\mathbf{B}$ is a definite positive matrix (rank $\mathbf{B} = n$).

Note that $\mathcal{A}^{-1}$ is compact so $D(\mathcal{A})$ is compactly embedded into $\mathcal{H}$. Thus we conclude that the spectrum of the operator $\mathcal{A}$ consists entirely of isolated eigenvalues. Based on this, it can be proved that $i\mathbb{R} \cap \sigma(\mathcal{A}) = \emptyset$, which is equivalent to $i\mathbb{R} \subset \varrho(\mathcal{A})$.

On the other hand, the resolvent equation can be written as

$$i\lambda\mathbf{U} - \mathcal{A}\mathbf{U} = \mathbf{F}, \tag{9}$$

where $\mathbf{U} = (U, V) \in D(\mathcal{A})$, $\mathbf{F} = (F, G) \in \mathbb{H}$ and $\lambda \in \mathbb{R}$. Taking the inner product in $\mathbb{H}$ and considering the real part we obtain

$$V(l)^*\mathbf{B}V(l) = \text{Re }(\mathbf{U}, \mathbf{F})_{\mathbb{H}} \qquad \Longrightarrow \qquad |V(l)|^2 \le C||\mathbf{U}||_{\mathbb{H}}||\mathbf{F}||_{\mathbb{H}}. \tag{10}$$

Theorem 3.1. *Suppose that $\mathbf{B}$ is a definite positive matrix, then the semigroup $\mathcal{S}_{\mathcal{A}}(t)$ is exponentially stable.*

Proof. We will show that

$$\limsup_{|\lambda|\to+\infty} ||(i\lambda I - \mathcal{A})^{-1}||_{\mathcal{L}(\mathcal{H})} < \infty.$$

Using (9) and (10) we have

$$|U_x(l)|^2 = |\mathbf{A}^{-1}\mathbf{B}V(l)|^2 \le C||\mathbf{U}||_{\mathbb{H}}||\mathbf{F}||_{\mathbb{H}} \tag{11}$$

Let $\mathbf{Q}$ be a real symmetric matrix. Note that for $W \in \mathcal{H}^1$, we have

$$\int_0^l x \frac{d}{dx} (W^* \mathbf{Q} W)\, dx = 2 \text{ Re } \int_0^l x W^* \mathbf{Q} W_x \; dx$$

and integrating by parts

$$2 \text{ Re} \int_0^l x W^* \mathbf{Q} W_x \; dx \;=\; l W^*(l) \mathbf{Q} W(l) \;-\; \int_0^l W^* \mathbf{Q} W \; dx. \tag{12}$$

From (9), we have

$$i\lambda V \;=\; \mathbf{A} U_{xx} + G,$$

and multiplying by $x {U_x}^*$ we have

$$-\int_0^l x {V_x}^* V \; dx \;=\; \int_0^l x {U_x}^* \mathbf{A} U_{xx} \; dx + \int_0^l x {U_x}^* (G + F_x) \; dx.$$

Taking the real part and applying (12), we obtain

$$\int_0^l \left(|V|^2 + U_x^* \mathbf{A} U_x\right) dx \;=\; l \left(|V(l)|^2 + U_x(l)^* \mathbf{A} U_x(l)\right) + R,$$

where R is a number satisfying $|R| \leq C ||\mathbf{U}||_{\mathbb{H}} ||\mathbf{F}||_{\mathbb{H}}$. Using (10) and (11) we obtain $||\mathbf{U}||_{\mathbb{H}} \leq C ||\mathbf{F}||_{\mathbb{H}}$ and our conclusion follows from Theorem 2.2.

□

4 Numerical Experiments

The goal of this section is to show numerical experiments in order to exemplify the concepts of exponential stability obtained in the previous sections.

As described above we are considering two problems, namely, mixture of elastic materials with friccional damping (1) and mixture of elastic materials with boundary fricccional damping (3)–(4). From the numerical point of view, the first problem is straigthforward. On the other hand, the second problem presents extra difficulties because of the boundary frictional condition.

Problem I: Here, we are interested in solving

$$\begin{cases} \mathbf{R} U_{tt} - \mathbf{A} U_{xx} + \mathbf{B} U_t = 0, \\ U(0,t) = 0 \;\text{ and }\; U(l,t) = 0, \quad t \in \mathbb{R}^+, \\ U(x,0) = U_0(x), \;\; U_t(x,0) = U_1(x), \end{cases} \tag{13}$$

where $U_0(x) = 0$ and $U_1(x) = (u_1^1(x), u_1^2(x), u_1^3(x))^T$ is given by:

$$u_1^i(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 0.4 \\ 10i(x - 0.4) & \text{if } 0.4 \leq x \leq 0.5 \\ 10i(0.6 - x) & \text{if } 0.5 \leq x \leq 0.6 \\ 0 & \text{if } 0.6 \leq x \leq 1 \end{cases}. \tag{14}$$

The problem above is a hyperbolic system with a frictional term. Theorem 1.1 says $\mathcal{S}_{\mathcal{A}}(t)$, the semigroup associated to (13), is exponentially stable if and only if $\mathbf{R}^{-1}\mathbf{A}$ and $\mathbf{B}$ are observable matrices.

We divide this problem in two cases. The first one, we present a case where $\mathbf{R}^{-1}\mathbf{A}$ and $\mathbf{B}$ are observable matrices, thus by Theorem 1.1 we expect an exponentially stable semigroup. For this case consider,

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{15}$$

In the second case, $\mathbf{R}^{-1}\mathbf{A}$ and $\mathbf{B}$ are not observable matrices, thus we do not expect an exponentially stable semigroup. For this case consider

$$A = \begin{pmatrix} 3 & 3 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{16}$$

Note that for both cases the frictional term is the same, only the propagation part is changed. To evaluate if a system is exponentially stable we need to analyze an energy functional associated to the system, given by equation (6).

Problem (13) is discretized using finite differences, more specifically we used a central-space scheme, and a central-time scheme for the second derivatives and a backward-time scheme for the first temporal derivative. The backward-time scheme was taken because $\mathbf{B}$ is a positive semidefinite matrix, which means any other discretization would require the calculation of the inverse of $\mathbf{B}$ or to solve a possible rank-deficient system.

Figure 1a presents the energy curves for cases (15) and (16). Case (15) is exponentially stable (blue curve), while case (16) is not exponentially stable (green curve).

Problem II: Here, we are interested in solving

$$\begin{cases} \mathbf{R}U_{tt} - \mathbf{A}U_{xx} = 0, \\ U(0,t) = 0 \quad \text{and} \quad \mathbf{A}U_x(l,t) + \mathbf{B}U_t(l,t) = 0, \quad t \in \mathbb{R}^+. \\ U(x,0) = U_0(x), \quad U_t(x,0) = U_1(x). \end{cases} \tag{17}$$

where $U_0(x) = 0$ and $U_1(x) = (u_1^1(x), u_1^2(x), u_1^3(x))^T$ is given by (14). Theorem 3.1 says, if $\mathbf{B}$ is full rank, then the semigroup $\mathcal{S}_{\mathcal{A}}(t)$ is exponentially stable.

For this problem we consider $\mathbf{R} = \mathbf{A} = \mathbf{I}$, and two cases for $\mathbf{B}$, called $\mathbf{B_1}$ and $\mathbf{B_2}$, given by

$$\mathbf{B_1} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B_2} = \begin{pmatrix} 6 & 0 & -2 \\ 0 & 3 & -2 \\ -2 & -2 & 2 \end{pmatrix}. \tag{18}$$

The main objective of this experiment is to show that under certain conditions ($\mathbf{B}$ is full rank) the boundary frictional damping is enough to guarantee the system is exponetially stable. For this reason, we present two different cases where the only difference is in the boundary condition. Based on Theorem 3.1, we expect the system with $\mathbf{B_1}$ (full rank) is exponentially stable, while the system with $\mathbf{B_2}$ (rank deficient) would not be exponentially stable.

Problem (17) is discretized using finite differences, more specifically we used a central-space-central-time scheme. The challenge here is the discretization of the boundary condition. Besides, consider $\mathbf{B}$ positive semidefinite, according to the well-posedeness of the problem. It implies that a numerical scheme must not take into account the inversibility of $\mathbf{B}$.

Thus,

$$\mathbf{A}U_x(l,t) + \mathbf{B}U_t(l,t) = 0 \qquad \Longrightarrow \qquad U_x(l,t) = -\mathbf{A}^{-1}\mathbf{B}U_t(l,t)$$

Applying a backward-time-backward-space scheme, it yields:

$$u_M^{n+1} = (I + \beta\mathbf{A}^{-1}\mathbf{B})^{-1}(u_{M-1}^{n+1} + \beta\mathbf{A}^{-1}\mathbf{B}u_M^n). \tag{19}$$

where $\beta = \frac{\Delta x}{\Delta t}$, n is related to the temporal discretization and M is the space discretization at the boundary. This scheme is valid if $(I+\beta\mathbf{A}^{-1}\mathbf{B})^{-1}$ exists. Such inverse is guaranteed by a well-known linear algebra result [3].

Figure 1b presents the energy curves for cases (18). The $\mathbf{B_1}$ case is exponentially stable (blue curve), while the $\mathbf{B_2}$ case is not exponentially stable (green curve).

In the first numerical experiment, the dissipation matrix $\mathbf{B}$ (rank defficient) is applied over the interval $[0,l]$. It exemplifies the relation of the exponential stability and the observability of the matrices $\mathbf{R}^{-1}\mathbf{A}$ and $\mathbf{B}$. In the second numerical experiment, the dissipation is located at the extreme $x = l$, we verified the $\mathbf{B}$ must be full rank. The outcome of the numerical experiments agree with Theorem 1.1 and Theorem 3.1. We conclude that the rank of dissipation influences the system stability and when it is restricted to the boundary of the interval, the dissipation must be full rank.

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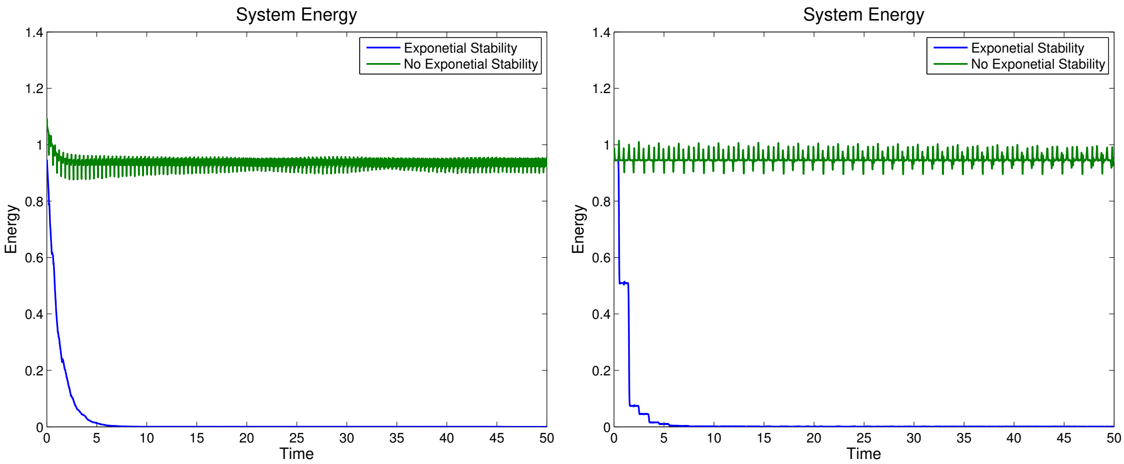

(a) Energy curves of problem (13) (b) Energy curves of problem (17)

Figure 1: Energy curves given by (6).

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