Proceeding Series of the Brazilian Society of Applied and Computational Mathematics, v. 6, n. 1, 2018.

Trabalho apresentado no XXXVII CNMAC, S.J. dos Campos - SP, 2017.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

Some Theoretical Aspects of max-C and min-D Projection Fuzzy Autoassociative Memories

Alex S. dos Santos[1]
Exact and Technology Sciences Center-CETEC, UFRB, Cruz das Almas, BA, Brazil
Marcos Eduardo do Valle[2]
Department of Applied Mathematics, UNICAMP, Campinas, SP, Brazil

Abstract. Autoassociative memories (AMs) are models inspired by the human ability to store and retrieve information by association. A fuzzy associative memory is used for the storage and recall of fuzzy sets. Inspired by the fuzzy morphological associative memories, we recently introduced the class of max-C projection fuzzy autoassociative memories (max-C PFAMs). In few words, a max-C PFAM projects the input vector into the set of all max-C combinations of the stored vectors. In this paper, we present its dual version, the class of min-D PFAMs, which projects an input into the set of all min-D combinations. In this paper we also address some theoretical issues of the two PFAM models.

Keywords. Associative memory, fuzzy set, mathematical morphology, adjunction.

1 Introduction

An autoassociative memory (AM) is an input-output system that allows for the storage and recall of a finite set of information [7]. Such system is inspired by the human ability to store and retrieve information by association. Such as the human brain, an AM memory is expected to retrieve a stored information upon presentation of a partial or corrupted version of an stored item.

Mathematically, an AM is formulated as follows: Given a set $\mathcal{A} = \{\mathbf{a}^1, \ldots, \mathbf{a}^k\}$, called the fundamental memory set, an AM is an application $\mathcal{M}$ such that $\mathcal{M}(\mathbf{a}^\xi) = \mathbf{a}^\xi$ holds as true as possible for $\xi \in \mathcal{K} = \{1, 2, \ldots, k\}$ [7]. Furthermore, the application $\mathcal{M}$ should present some noise tolerance, i.e., we expect $\mathcal{M}(\widetilde{\mathbf{a}}^\xi) = \mathbf{a}^\xi$ if $\widetilde{\mathbf{a}}^\xi$ is a corrupted or partial version of $\mathbf{a}^\xi$.

The AMs achieved further notoriety in the scientific community after the work of Hopfield in the early 1980s [7]. In few words, Hopfield showed that a discrete nonlinear dynamic system can be designed to implement an AM for the storage and recall of binary patterns.

The first AM model designed for the storage and recall of fuzzy set have been introduced by Kosko in 1992 [8]. Briefly, Kosko's fuzzy associative memory is described in

[1]assantos@ufrb.edu.br
[2]valle@ime.unicamp.br

terms of either the max-minimum or max-product compositions. Despite the successful applications of the FAMs of Kosko, this associative memory model suffers from a very low storage capacity [3, 8]. As a consequence, many other FAM models have been developed subsequently [6, 13–15]. Applications of the FAMs include classification [6], times-series prediction [15], and image reconstruction [11, 13, 15].

Inspired by the max-plus projection autoassociative morphological memories (max-plus PAMMs) introduced by Valle [12], we proposed recently the class of max-C projection fuzzy autoassociative memories (max-C PFAMs) [10]. In few words, a max-C PFAM projects the input pattern into set whose elements are max-C combinations of the items stored. In this work, we present a dual model, called min-D PFAM, in which the input is projected into the set of all min-D combinations of the stored items. Furthermore, we present some theoretical results concerning both max-C and min-D PFAMs.

The paper is organized as follows. We first review some basic concepts of fuzzy logic in the next section. The max-C and min-D PFAMs are discussed in Section 3. In Section 3 we also present some theoretical results and address the duality relationship between the two PFAMs. We finish the paper with the concluding remarks.

2 Basic Concepts of Fuzzy Logical Operators

The AM models considered in this paper are based in fuzzy logic operations, namely, fuzzy conjunction, fuzzy disjunction, fuzzy implication, fuzzy coimplication, and fuzzy negation. For a more detailed treatment on these operators, the reader is referred to [1,4,9]. We would like to point out that, in this paper, the symbols "$\vee$" and "$\wedge$" are used to represent respectively the supremum (maximum) and infimum (minimum) operations.

Definition 2.1 (Fuzzy conjunction). *A fuzzy conjunction $C : [0,1] \times [0,1] \longrightarrow [0,1]$ is an increasing operator that satisfies $C(0,0) = C(0,1) = C(1,0) = 0$ and $C(1,1) = 1$.*

Definition 2.2 (Fuzzy disjunction). *A fuzzy disjunction $D : [0,1] \times [0,1] \longrightarrow [0,1]$ is an increasing operator that satisfies $D(1,1) = D(0,1) = D(1,0) = 1$ and $D(0,0) = 0$.*

Definition 2.3 (Fuzzy Implication). *A fuzzy implication is an operator $I : [0,1] \times [0,1] \longrightarrow [0,1]$ decreasing in the first argument, increasing in the second argument, which satisfies $I(0,0) = I(0,0) = 1$ and $I(1,0) = 0$.*

Definition 2.4 (Fuzzy coimplication). *A fuzzy coimplication is an operator $J : [0,1] \times [0,1] \longrightarrow [0,1]$ decreasing in the first argument, increasing in the second argument, which satisfies $J(0,0) = J(1,1) = 0$ and $J(0,1) = 1$.*

A fuzzy conjunction and a fuzzy implication, as well as a fuzzy disjunction and a fuzzy coimplication, can be related through of a fundamental concept of the mathematical morphology called adjunction [5].

Definition 2.5 (Adjunction). *A fuzzy implication I and a fuzzy conjunction C form an adjunction if the following relations hold true*

$$C(x,y) \leq z \Longleftrightarrow x \leq I(y,z), \quad \forall x,y,z \in [0,1]. \tag{1}$$

Similarly, a fuzzy disjunction D and fuzzy coimplication J form an adjunction if

$$D(x,y) \geq z \Longleftrightarrow x \geq J(y,z), \quad \forall x,y,z \in [0,1]. \tag{2}$$

Definition 2.6 (Fuzzy negation)**.** *A decreasing operator $\eta : [0,1] \longrightarrow [0,1]$ is a fuzzy negation if $\eta(0) = 1$ and $\eta(1) = 0$. Moreover, if $\eta(\eta(x)) = x$ for all $x \in [0,1]$, we say that η is a strong fuzzy negation.*

The fuzzy conjunction C can be connected to a fuzzy disjunction D by mean of the duality relationship with respect to a fuzzy negation η as follows:

$$D(x,y) = \eta\Big(C\big(\eta(x),\eta(y)\big)\Big) \quad \text{or} \quad C(x,y) = \eta\Big(D\big(\eta(x),\eta(y)\big)\Big),\ \forall x,y \in [0,1] \tag{3}$$

In a similar manner, a fuzzy coimplication J is the dual operator of a fuzzy implication I with relation the fuzzy negation η if and only if

$$J(x,y) = \eta\Big(I\big(\eta(x),\eta(y)\big)\Big) \quad \text{or} \quad I(x,y) = \eta\Big(J\big(\eta(x),\eta(y)\big)\Big),\ \forall x,y \in [0,1] \tag{4}$$

In analogy to the concept of linear combination, we say that $\mathbf{z} \in [0,1]^n$ is a max-C combinations of the vectors belonging to the finite set $\mathcal{A} = \{\mathbf{a}^1, \ldots, \mathbf{a}^k\} \subseteq [0,1]^n$ if

$$\mathbf{z} = \bigvee_{\xi=1}^{k} C(\lambda_\xi, \mathbf{a}^\xi) \quad \Longleftrightarrow \quad z_i = \bigvee_{\xi=1}^{k} C(\lambda_\xi, a_i^\xi),\ \forall i = 1,\ldots,n, \tag{5}$$

where $\lambda_\xi \in [0,1]$ for all $\xi = 1,\ldots,k$. Similarly, a min-D combination of the vectors of $\mathcal{A}$ is given by

$$\mathbf{y} = \bigwedge_{\xi=1}^{k} D(\theta_\xi, \mathbf{a}^\xi) \quad \Longleftrightarrow \quad y_i = \bigwedge_{\xi=1}^{k} D(\theta_\xi, a_i^\xi),\ \forall i = 1,\ldots,n, \tag{6}$$

where $\theta_\xi \in [0,1]$, for all $\xi = 1,\ldots,k$. The sets of all max-C combinations and min-D combinations of $\mathcal{A} = \{\mathbf{a}^1, \ldots, \mathbf{a}^k\} \subseteq [0,1]^n$ are respectively defined by

$$\mathcal{C}(\mathcal{A}) = \left\{ \mathbf{z} = \bigvee_{\xi=1}^{k} C(\lambda_\xi, \mathbf{a}^\xi) : \lambda_\xi \in [0,1] \right\} \text{ and } \mathcal{D}(\mathcal{A}) = \left\{ \mathbf{z} = \bigwedge_{\xi=1}^{k} D(\theta_\xi, \mathbf{a}^\xi) : \theta_\xi \in [0,1] \right\}. \tag{7}$$

3 Max-C and min-D PFAMs

First of all, recall that a fuzzy set $\mathbf{x}$ in a finite universe $U = \{u_1, u_2, \ldots, u_n\}$ can be identified with a vector $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T \in [0,1]^n$, where the component $x_j = \mathbf{x}(u_j)$ denotes the degree of pertinence of u_j in the fuzzy set $\mathbf{x}$ [9]. We denote the family of all fuzzy sets on U by $\mathcal{F}(U)$. Now, an application $\mathcal{M}$ is a fuzzy associative memory (FAM) if it is designed for the storage and recall fuzzy set, i.e., $\mathcal{M} : \mathcal{F}(U) \longrightarrow \mathcal{F}(U)$.

A max-C projection fuzzy autoassociative memory (max-C PFAM) is the mapping that projects an input $\mathbf{x}$ downright into the set of all max-C combinations of $\mathbf{a}^1, \ldots, \mathbf{a}^k$ [10]. In mathematical terms, given a fundamental memory set $\mathcal{A} = \{\mathbf{a}^1, \ldots, \mathbf{a}^k\}$, a max-$C$ PFAM $\mathcal{V} : [0,1]^n \to [0,1]^n$ is defined by

$$\mathcal{V}(\mathbf{x}) = \bigvee \{\mathbf{z} \in \mathcal{C}(\mathcal{A}) : \mathbf{z} \leq \mathbf{x}\}, \quad \forall \mathbf{x} \in [0,1]^n. \tag{8}$$

In a similar fashion, let us define a min-D PFAM as the mapping that projects an input $\mathbf{x}$ upright into the set of all min-D combinations of $\mathbf{a}^1, \ldots, \mathbf{a}^k$. In mathematical terms, given a fundamental memory set $\mathcal{A}$, a min-D PFAM $\mathcal{S} : [0,1]^n \to [0,1]^n$ is defined by

$$\mathcal{S}(\mathbf{x}) = \bigwedge \{\mathbf{y} \in \mathcal{D}(\mathcal{A}) : \mathbf{y} \geq \mathbf{x}\}, \quad \forall \mathbf{x} \in [0,1]^n. \tag{9}$$

The following theorem is a straightforward consequence of previous definition.

Theorem 3.1. *The max-C and min-D PFAMs given respectively by* (8) *and* (9) *satisfy the inequalities $\mathcal{V}(\mathbf{x}) \leq \mathbf{x} \leq \mathcal{S}(\mathbf{x})$ for any input pattern $\mathbf{x} \in [0,1]^n$.*

The following theorem shows that the max-C and min-D PFAMs have optimal absolute storage capacity if C and D have a left identity.

Theorem 3.2. *If the fuzzy conjunction C and fuzzy disjunction D have left identity, then the max-C and min-D PFAMs satisfy the equations $\mathcal{V}(\mathbf{a}^\xi) = \mathbf{a}^\xi = \mathcal{S}(\mathbf{a}^\xi)$, for all $\xi \in \mathcal{K}$.*

Assume that the fuzzy conjunction C and the fuzzy disjunction D have a left identity. From Theorem 3.1, a max-C PFAM is able retrieve a fundamental memory $\mathbf{a}^\xi$ if and only if the input $\mathbf{x}$ satisfies $\mathbf{x} \geq \mathbf{a}^\xi$. Also, a min-$D$ PFAM is able to recall $\mathbf{a}^\xi$ if and only if $\mathbf{x} \leq \mathbf{a}^\xi$. We say that a distorted version $\mathbf{x}$ of the original vector $\mathbf{a}^\xi$ has undergone a dilative change if $\mathbf{x} \geq \mathbf{a}^\xi$. Similarly, we say that a corrupted version $\mathbf{x}$ of $\mathbf{a}^\xi$ has undergone an erosive change if $\mathbf{x} \leq \mathbf{a}^\xi$. Using this terminology, we can assert that the max-C PFAM is robust in the presence of dilative noise but it is not effective in the presence of erosive noise. Dually, a min-D PFAM is robust in the presence of erosive noise but it is not effective in the presence of dilative changes.

The next theorem provides effective formulas for the implementation of the max-C and min-D PFAMs.

Theorem 3.3. *Consider a fundamental memory set $\mathcal{A} = \{\mathbf{a}^1, \ldots, \mathbf{a}^k\} \subseteq [0,1]^n$. Let a fuzzy implication I and a fuzzy conjunction C form an adjunction. For any input $\mathbf{x} \in [0,1]^n$, the max-C PFAM $\mathcal{V}$ satisfies*

$$\mathcal{V}(\mathbf{x}) = \bigvee_{\xi=1}^{k} C(\lambda_\xi, \mathbf{a}^\xi), \quad \textit{where} \quad \lambda_\xi = \bigwedge_{j=1}^{n} I(a_j^\xi, x_j). \tag{10}$$

Dually, let a fuzzy coimplication J and a fuzzy disjunction D form an adjunction. For any input $\mathbf{x} \in [0,1]^n$, the output of min-D PFAM $\mathcal{S}$ can be computed by

$$\mathcal{S}(\mathbf{x}) = \bigwedge_{\xi=1}^{k} D(\theta_\xi, \mathbf{a}^\xi), \quad \textit{where} \quad \theta_\xi = \bigvee_{j=1}^{n} J(a_j^\xi, x_j). \tag{11}$$

Proof. We are going to proof only the first part of Theorem 3.3. The second part can be derived in a similar manner. Let $\mathbf{z} \in \mathcal{C}(\mathcal{A})$ be a max-C combination of $\mathbf{a}^1, \ldots, \mathbf{a}^k$ and consider the set of indexes $\mathcal{N} = \{1, \ldots, n\}$ and $\mathcal{K} = \{1, \ldots, k\}$. Since the fuzzy implication I and the fuzzy conjunction C form an adjunction, we have:

$$\begin{aligned}
\mathbf{z} \leq \mathbf{x} \iff & \bigvee_{\xi=1}^{k} C(\lambda_\xi, a_j^\xi) \leq x_j, \ \forall j \in \mathcal{N} \\
\iff & C(\lambda_\xi, a_j^\xi) \leq x_j, \ \forall k \in \mathcal{K}, \forall j \in \mathcal{N} \\
\iff & \lambda_\xi \leq I(a_j^\xi, x_j), \ \forall j \in \mathcal{N}, \ \forall \xi \in \mathcal{K} \\
\iff & \lambda_\xi \leq \bigwedge_{j=1}^{n} I(a_j^\xi, x_j), \ \forall \xi \in \mathcal{K}.
\end{aligned}$$

Thus, the largest max-C combination $\mathbf{z} = \bigvee_{\xi=1}^{k} C(\lambda_\xi, \mathbf{a}^\xi)$ such that $\mathbf{z} \leq \mathbf{x}$ is obtained by considering $\lambda_\xi = \bigwedge_{j=1}^{n} I(a_j^\xi, x_j)$ for all $\xi \in \mathcal{K}$. □

Remark 3.1. *We would like to point out that the parameter λ_ξ measures the degree of inclusion of $\mathbf{a}^\xi$ in $\mathbf{x}$ in the sense of Bandler-Kohout [2].*

Both max-C and min-D PFAMs belong to the broad class of fuzzy morphological associative memories (FMAMs) because they perform elementary operations of mathematical morphology [14]. It turns out from mathematical morphology that, given an FMAM model $\mathcal{W}$, we can construct another FMAM $\mathcal{W}^*$, called the negation of $\mathcal{W}$, using a strong fuzzy negation. Precisely, the negation $\mathcal{W}^*$ is defined as follows where the fuzzy negation η is applied in a component-wise manner:

$$\mathcal{W}^*(\mathbf{x}) = \eta\Big(\mathcal{W}\big(\eta(\mathbf{x})\big)\Big), \quad \forall \mathbf{x} \in [0,1]^n. \tag{12}$$

The next theorem shows that the negation of a min-D PFAM is a max-C PFAM designed for the storage of the negation of the fundamental memories, and vice-versa.

Theorem 3.4. *Let a fuzzy conjunction C be connected to a fuzzy disjunction D by means of a strong fuzzy negation η. Given a fundamental memory set $\mathcal{A} = \{\mathbf{a}^1, \ldots, \mathbf{a}^k\} \subseteq [0,1]^n$, define $\mathcal{B} = \{\mathbf{b}^1, \ldots, \mathbf{b}^k\}$ by setting $b_i^\xi = \eta(a_i^\xi)$ for all $i = 1, \ldots, n$ and $\xi = 1, \ldots, k$. The negation $\mathcal{S}^*$ of the min-D PFAM $\mathcal{S}$ designed for the storage of $\mathbf{a}^1, \ldots, \mathbf{a}^k$ is the max-C PFAM designed for the storage of $\mathbf{b}^1, \ldots, \mathbf{b}^k$, that is,*

$$\mathcal{S}^*(\mathbf{x}) = \bigvee_{\xi=1}^{k} C(\lambda_\xi^*, \mathbf{b}^\xi), \quad \textit{where} \quad \lambda_\xi^* = \bigwedge_{j=1}^{n} I(b_j^\xi, x_j). \tag{13}$$

Analogously, the negation $\mathcal{V}^*$ of the max-C PFAM $\mathcal{V}$ designed for the storage of $\mathbf{a}^1, \ldots, \mathbf{a}^k$ is the min-D PFAM defined by

$$\mathcal{V}^*(\mathbf{x}) = \bigwedge_{\xi=1}^{k} D(\theta_\xi^*, \mathbf{b}^\xi), \quad \textit{where} \quad \theta_\xi^* = \bigvee_{j=1}^{n} J(b_j^\xi, x_j). \tag{14}$$

Proof. Let us only show (13). The second part of the theorem is derived in a similar manner. Recalling that a strong negation is a decreasing operator and, thus, the negation of the minimum is the maximum of the negations, we conclude respectively from (12), (11), and (3) that

$$\mathcal{S}^*(\mathbf{x}) = \eta\Big(\mathcal{S}\big(\eta(\mathbf{x})\big)\Big) = \eta\left(\bigwedge_{j=1}^{n} D(\theta_\xi, \mathbf{a}^\xi)\right) = \bigvee_{\xi=1}^{k} \eta\big(D(\theta_\xi, \mathbf{a}^\xi)\big) = \bigvee_{\xi=1}^{k} C(\lambda_\xi^*, \mathbf{b}^\xi),$$

where $\lambda_\xi^* = \eta(\theta_\xi)$ satisfies the following identities

$$\lambda_\xi^* = \eta\left(\bigvee_{j=1}^{n} J\big(a_j^\xi, \eta(x_j)\big)\right) = \bigwedge_{j=1}^{n} \eta\left(J\big(a_j^\xi, \eta(x_j)\big)\right) = \bigwedge_{j=1}^{n} I\Big(\eta\big(a_j^\xi\big), x_j\Big) = \bigwedge_{j=1}^{n} I\Big(b_j^\xi, x_j\Big).$$

From (10), we conclude that $\mathcal{S}^*$ is the max-C PFAM designed for the storage of $\mathbf{b}^1, \ldots, \mathbf{b}^k$. □

4 Concluding Remarks

The max-C PFAMs are associative memories designed for the storage and recall of vectors in the hypercube $[0,1]^n$ [10]. In this paper, we introduced the class of min-D projection fuzzy autoassociative memories (min-D PFAMs) in a manner similar to the max-C PFAM models. Then, we provided some theoretical results concerning the storage capacity and noise tolerance of the max-C and min-D PFAMs. In particular, we pointed out that we can store as many vectors as desired in a max-C or min-D PFAM if the fuzzy conjunction or fuzzy disjunction has a left identity. Also, we pointed out that these memories models are robust in the presence of either dilative or erosive noise but they are inefficient if the input is corrupted by both types of noise. We provided an efficient method to computed the output of max-C and min-D PFAM models. Finally, we shows that the negation of a min-D PFAM is a max-C PFAM designed for the storage of the negation of the fundamental memories, and vice-versa.

Acknowledgments

This work was supported in part by Capes, Programa de Formação Docente, and CNPq under grant no 305486/2014-4.

References

[1] M. Baczyński and B. Jayaram. *Fuzzy Implications from Fuzzy Logic Operations*, pages 39–107. Springer Berlin Heidelberg, Berlin, Heidelberg, 2008.

[2] W. Bandler and L. Kohout. Fuzzy power sets and fuzzy implication operators. *Fuzzy Sets and Systems*, 4(1):13–30, July 1980. DOI:10.1016/0165-0114(80)90060-3.

[3] F. Chung and T. Lee. On Fuzzy Associative Memory with Multiple-Rule Storage Capacity. *IEEE Transactions on Fuzzy Systems*, 4(3):375–384, 1996.

[4] B. De Baets. Coimplicators, the forgotten connectives. *Trata Mountains Mathematical Publications*, 12:229–240, 1997. Fulltext available at: http://www.mat.savba.sk/tatra/.

[5] T. Deng and H. Heijmans. Grey-scale morphology based on fuzzy logic. *Journal of Mathematical Imaging and Vision*, 16(2):155–171, 2002. DOI:10.1023/A:1013999431844.

[6] E. Esmi, P. Sussner, H. Bustince, and J. Fernández. Theta-fuzzy associative memories (Theta-FAMs). *IEEE Transactions on Fuzzy Systems*, 23(2):313–326, April 2015.

[7] M. H. Hassoun and P. B. Watta. Associative Memory Networks. In E. Fiesler and R. Beale, editors, *Handbook of Neural Computation*, pages C1.3:1–C1.3:14. Oxford University Press, 1997.

[8] B. Kosko. *Neural Networks and Fuzzy Systems: A Dynamical Systems Approach to Machine Intelligence*. Prentice Hall, Englewood Cliffs, NJ, 1992. ISBN: 978-0136114352.

[9] W. Pedrycz and F. Gomide. *Fuzzy Systems Engineering: Toward Human-Centric Computing*. Wiley-IEEE Press, New York, 2007. ISBN: 978-0-471-78857-7.

[10] A. S. Santos and M. E. Valle. Uma introdução às memórias autoassociativas fuzzy de projeções max-C. In *Recentes Avanços em Sistemas Fuzzy. Sociedade Brasileira de Matemática Aplicada e Computacional, Volume 1* , São Carlos, Brasil, Novembro 2016. ISBN: 78-85-8215-079-5.

[11] P. Sussner and M. E. Valle. Implicative Fuzzy Associative Memories. *IEEE Transactions on Fuzzy Systems*, 14(6):793–807, Dec. 2006. DOI: 10.1109/TFUZZ.2006.879968.

[12] M. E. Valle. An Introduction to the Max-plus Projection Autoassociative Morphological Memory and Some of Its Variations. In *International Conference on Fuzzy Systems (FUZZ-IEEE 2014)*, pages 53 – 60, Beijing , China, July 2014.

[13] M. E. Valle and A. C. de Souza. On the recall capability of recurrent exponential fuzzy associative memories based on similarity measures. *Mathware and Soft Computing Magazine*, 22:33–39, 2015. ISBN:11134-5632.

[14] M. E. Valle and P. Sussner. A General Framework for Fuzzy Morphological Associative Memories. *Fuzzy Sets and Systems*, 159(7):747–768, 2008. DOI:10.1016/j.fss.2007.10.010.

[15] M. E. Valle and P. Sussner. Storage and Recall Capabilities of Fuzzy Morphological Associative Memories with Adjunction-Based Learning. *Neural Networks*, 24(1):75–90, Jan. 2011. DOI:10.1016/j.neunet.2010.08.013.

