Proceeding Series of the Brazilian Society of Applied and Computational Mathematics, v. 6, n. 1, 2018.

Trabalho apresentado no XXXVII CNMAC, S.J. dos Campos - SP, 2017.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

Noise Reduction on Numerical Solutions of Partial Differential Equations using Fuzzy Transform

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Abstract. Fuzzy Transform (F-transform) has been introduced as an approximation method which encompasses both classical transforms as well as approximation methods studied in fuzzy modelling and fuzzy control. It has been proved that, under some conditions, F-transform can remove a periodical noise and it can significantly reduce random noise. In this work we apply the F-transform methodology on the study of numerical solutions of partial differential equations with noisy initial conditions.

Keywords. Numerical Analysis, FD schemes, Fuzzy Transform, Noise Reduction.

1 Introduction

Fuzzy Transform (F-transform) has been introduced as an approximation method which encompasses both classical transforms and approximation methods studied in fuzzy modeling and fuzzy control, see [3].

Many valuable properties of this method has been shown and, as consequence many applications have been published, from geology [2] to electrical engineering [6], including image processing [3] and differential equations [7]. As in other approximation techniques, it can be shown that if an original function is replaced by an approximation model, then a certain simplification of complex computations could be achieved, the F-transform methodology has been demonstrated to be a robust approximation method. It has been proved that, under some conditions, it can remove periodical noise and it can significantly reduce random noise [4, 7].

The main idea consists in the replacement of a continuous function on a real compact interval by its discrete representation (using the direct F-transform). According to the specific situation, computations or calculus are made on discrete representation of the function and then transformed back to the space of continuous functions (using the inverse F-transform). The result obtained by applying both F-transforms is a good simplified approximation of the original function.

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For several reasons, in practice, noisy initial conditions are presented when solving partial differential equations (PDEs). F-transform methodology is a powerful tool for noise reduction on numerical solutions of PDEs. It can be applied to a variety of PDEs, including classical linear equations, like heat equation and wave equation, and non-linear equations, like Burgers' equation or hyperbolic conservation laws. The concepts of the approximation theory behind F-transform and how it handles noise is presented in this text. To avoid a tedious presentation of the problem, only the wave equation case is shown.

2 The Fuzzy Transform and its properties

Fuzzy Transform (F-transform) has been introduced as an approximation method which encompasses both classical transforms and approximation methods studied in fuzzy modeling and fuzzy control, see [3]. In this section we introduce the definition of F-transform and its inverse, the definitions presented here can be encountered in [3]. Consider the real-valued compact interval $[a, b]$ as an universe. The first definition we introduce is about the basic functions,

Definition 2.1. *Let $x_i = a + h(i-1)$ be nodes on $[a, b]$ where $h = (b-a)/(n-1)$, $n \geq 2$ and $i = 1, ..., n$. We say that fuzzy sets $A_1, ..., A_n$ identified by the membership functions $A_1(x), ..., A_n(x)$ defined on $[a, b]$ form an uniform fuzzy partition of $[a, b]$ if:*

- $A_i : [a, b] \to [0, 1]$, $A_i(x_i) = 1$ *and* $\displaystyle\sum_{i=1}^{n} A_i(x) = 1$, *for all* $x \in [a, b]$,
- $A_i(x) = 0$ *if* $x \notin (x_{i-1}, x_{i+1})$ *where* $x_0 = a, x_{n+1} = b$,
- $A_i(x)$ *is continuous, strictly increases on* $[x_{i-1}, x_i]$ *and strictly decreases on* $[x_i, x_{i+1}]$,,
- $A_i(x_i - x) = A_i(x_i + x)$, *for all* $x \in [0, h]$, $i = 2, ..., n-1, n > 2$,
- $A_{i+1}(x) = A_i(x - h)$, *for all* $x \in [a + h, b], i = 2, ..., n-2, n > 2$.

The membership functions are called basic functions.

A typical example of basic functions are uniform triangular functions. The selection of the membership functions can be defined for example on smoothness. In some cases, extended fuzzy partitions might be considered [5]. Now, the notion of F-transform and its inverse in one variable will be defined, the generalization to two or more variables is straightforward.

Definition 2.2. *Let $f(x)$ be a continuous function on $[a, b]$ and $A_1(x), ..., A_n(x)$ be basic functions determining an uniform fuzzy partition of $[a, b]$. The n-tuple of real numbers $[F_1, ..., F_n]$ such that*

$$F_i = \frac{\int_a^b f(x) A_i(x) dx}{\int_a^b A_i(x) dx}, \quad i = 1, ..., n, \tag{1}$$

will be called the F-transform of f with respect to the given basic functions [7].

Note that, on the definition above, the F-transform is a linear transformation from $C[a,b]$ to $\mathbb{R}^n$.

Definition 2.3. *Let $A_1, ..., A_n$ be basic functions and let $F_n[f] = [F_1, ..., F_n]$ be the F-transform of f with respect to $A_1, ..., A_n$. The function*

$$f_n^F(x) = \sum_{i=1}^{n} A_i(x) F_i \tag{2}$$

will be called the inverse F-transform [7].

For our purposes we have to generalize the definitions above. So, we introduce the definition of a 2-dimensional F-transform and its inverse.

Definition 2.4. *Let $f(x,y)$ be an arbitrary continuous function on $\mathcal{D} = [a,b] \times [c,d]$ and let $A_1(x), ..., A_n(x)$ on $[a,b]$ and $B_1(y), ..., B_m(y)$ on $[c,d]$ form uniform fuzzy partitions, not necessarily the same. We say that a matrix $[F_{ij}]_{nm}$ of real numbers is the F-transform of $f(x,y)$ with respect to the given basic functions if*

$$F_{ij} = \frac{\int_c^d \int_a^b f(x,y) A_i(x) B_j(y) dx dy}{\int_c^d \int_a^b A_i(x) B_j(y) dx dy}, \quad i = 1, ..., n, \ j = 1, ..., m, \tag{3}$$

Moreover, let $[F_{ij}]_{nm}$ be the F-transform of a function f. Then the function

$$f_{n,m}^F(x,y) = \sum_{i=1}^{n} \sum_{j=1}^{m} A_i(x) B_j(y) F_{ij} \tag{4}$$

will be called the inverse F-transform. [7]

The purpose of this work is to show how F-transform impacts finite-difference (FD) solutions of certain classes of partial differential equations (PDEs) when the initial condition is contaminated by noise. Fuzzy transform works well in removing noise. An extensive work was done on [4] to identify what classes of noise can be removed using F-transforms.

The theorem below shows the approximation properties of the inverse F-transform with a given arbitrary precision [4].

Theorem 2.1. *Let $f \in C[a,b]$. Then, for any $\epsilon > 0$, there exists n_ϵ and a fuzzy partition $A_1, ..., A_n$ of $[a,b]$ such that for all $x \in [a,b]$*

$$|f(x) - f_{n_\epsilon}^F(x)| < \epsilon,$$

where $f_{n_\epsilon}^F(x)$ is the inverse F-transform of f with respect to the fuzzy partitions $A_1, ..., A_{n_\epsilon}$.

Besides that, the forward F-transform components can be seen as a minimum of a certain functional [3].

DOI: 10.5540/03.2018.006.01.0402

Theorem 2.2. *Let $f \in C[a,b]$ and $A_1, ..., A_n$ be basic functions which form a fuzzy partition of $[a,b]$. Then the kth component of the integral F-transform gives minimum to the functional*

$$\int_a^b (f(x) - y)^2 A_i(x)\ dx.$$

Theorems 2.1 and 2.2 provide the tools to understand the approximation properties of the F-transform. Besides, when to the original function, additive noise is present, F-transform methodology allows to attenuate and sometimes eliminate the additive noise presented. In the next section we will explore the behavior of the F-transform methodology when applied on FD-schemes to solve PDEs with noisy initial conditions.

The inverse F-transform properties of sampling and interpolation can be improved by using the so-called fuzzy projection [1]. The ideas of fuzzy projection will not be used in this study because it does not permit to created a transform methodology to apply on numerical solutions of PDEs.

3 Fuzzy transform and finite-difference schemes

F-transform can be used to solve PDEs numerically, specifically using finite-difference (FD) schemes. Moreover, F-transform approach, on FD schemes, works well when noise is presented on the external force factor [7]. In this work our approach is slightly different, we assume the initial condition is perturbed by noise.

For the rest of the text we assume that our functions have as many continuous derivatives as necessary to the calculations below are valid. Consider the class of differential equations

$$\sum_{n=1}^{N} \alpha_n \frac{\partial^n}{\partial t^n} u + \sum_{m=1}^{M} \beta_m \frac{\partial^m}{\partial x^m} f(u) = q(x,t), \tag{5}$$

where N, M, α_n and β_n are finite constants, and q is the external force. Special cases of equation (5) are the conservation laws: $\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} f(u) = 0$, wave equation: $\frac{\partial^2}{\partial t^2} u - \frac{\partial^2}{\partial x^2} u = 0$, Burgers' equation: $\frac{\partial}{\partial t} u - \frac{\partial}{\partial x} \left(\frac{u^2}{2} \right) = \epsilon \frac{\partial^2}{\partial x^2} u$, etc.

We show how to use, in details, F-transform to solve the wave equation. The calculation for the general case are quite tedious, the ideas presented for the wave equation is enough to understand the process. The derivation described below follows the ideas presented in [7]. Let us consider a set $D = [a,b] \times [0,T]$ and let $u(x,t)$ be a continuous solution of the following equation on D,

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = q(x,t), \tag{6}$$

with the following initial and boundary conditions

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x), \qquad u(a,t) = T_1(t), \qquad u(b,t) = T_2(t). \tag{7}$$

Applying the F-transform, equation (6) is turned into the following algebraic equation

$$U_{ij}^{xx} - \alpha U_{ij}^{tt} = Q_{ij}, \tag{8}$$

where U_{ij}^{xx}, U_{ij}^{tt} and Q_{ij} are the F-transform components of the functions $\dfrac{\partial^2 u}{\partial x^2}, \dfrac{\partial^2 u}{\partial t^2}$ and q respectively. Using central FD schemes to aproximate partial derivatives in left side of (6) we would have

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{u(x+h_x,t) - 2u(x,t) + u(x-h_x,t)}{h_x^2} + O(h_x^2) \\ \dfrac{\partial^2 u}{\partial t^2} = \dfrac{u(x,t+h_t) - 2u(x,t) + u(x,t-h_t)}{h_t^2} + O(h_t^2) \end{cases}. \tag{9}$$

We are going to use these expressions to estimate the F-transforms, U_{ij}^{xx} and U_{ij}^{tt}, in terms of a recurrence relation, but first, some calculations have to be done.

For $j = 2, ..., m-2$, we can consider the function $u(x, t+h_t)A_i(x)B_j(t)$ being defined on the entire interval $[0, T]$ – even if $u(x, t+h_t)$ it is not defined on $(T-h_t, T] = (t_m, t_{m+1}]$ – because $B_j(t)$ is zero on $(t_m, t_{m+1}]$, so we have that

$$\int_a^b \int_0^T u(x, t \pm h_t) A_i(x) B_j(t) dt dx = \int_a^b \int_0^T u(x,t) A_i(x) B_{j\pm1}(t) dt dx. \tag{10}$$

This last expression is valid also when $j = 1$ or $j = m-1$, because definition of uniform fuzzy partition. It is also easy to prove that for $j = 1, ..., m-1$.

$$\int_a^b \int_0^T A_i(x) B_j(t) dt dx = \int_a^b \int_0^T A_i(x) B_{j\pm1}(t) dt dx. \tag{11}$$

Now we can use these previous calculations to estimate U_{ij}^{tt} and U_{ij}^{xx} as follows:

$$U_{ij}^{tt} \approx \frac{1}{h_t^2}(U_{i(j+1)} - 2U_{ij} + U_{i(j-1)}) \qquad \text{and} \qquad U_{ij}^{xx} \approx \frac{1}{h_x^2}(U_{(i+1)j} - 2U_{ij} + U_{(i-1)j}). \tag{12}$$

Applying (12) to equation (8) we obtain the following recursive equation

$$U_{i(j+1)} = r^2 U_{(i-1)j} + 2(1-r^2)U_{ij} + r^2 U_{(i+1)j} - U_{i(j-1)} - h_t^2 Q_{ij} \tag{13}$$

where $r = h_t/h_x$ and $i = 1, ..., n,\ j = 1, .., m$. We still need to deal with initial and boundary conditions, for example, the F-transform of the condition $u(a,t) = T_1(t)$ yields

$$\begin{aligned} U_{1j} &= \frac{\int_a^b \int_0^T u(x,t) A_1(x) B_j(t) dt dx}{\int_a^b \int_0^T A_1(x) B_j(t) dt dx} = \frac{\int_{t_{j-1}}^{t_{j+1}} \int_a^{x_2} u(x,t) A_1(x) B_j(t) dx dt}{\int_{t_{j-1}}^{t_{j+1}} \int_a^{x_2} A_1(x) B_j(t) dx dt} \\ &\approx \frac{\int_{t_{j-1}}^{t_{j+1}} \int_a^{x_2} T_1(t_j) A_1(x) B_j(t) dx dt}{\int_{t_{j-1}}^{t_{j+1}} \int_a^{x_2} A_1(x) B_j(t) dx dt} = T_1(t_j) = T_1((j-1)h_t), \end{aligned} \tag{14}$$

and in the same way we obtain

$$U_{nj} = T_2((j-1)h_t), \qquad U_{i1} = f((i-1)h_x) \qquad U_{i2} = U_{i1} + h_t g((i-1)h_x).$$

Now, we have a complete description of the recursive equation for the F-transform U_{ij}. Note that this recursive equation is similar to the finite-difference recursive equation for analogous partial derivates, which means that in this case, our approximation process for the F-transform is mostly equivalent to the FD method, even if they are conceptually different. We can take advantage of this similarity and, in fact, we can say, for example, that for the numerical stability of algorithm (13), condition $0 < r \leq 1$ must be valid. We can say even more, using well-known techniques, it is easy to prove a similar result about convergence of (13), see [8]. After running (13), the result is the solution of the F-transformed equation, to return to the original domain an inverse F-transform must be applied.

4 Numerical tests

In this section we present some numerical tests to illustrate the previous results. For this purpose we solved, numerically, the homogeneous version of equation (6), with the following conditions:

$$u(x,0) = \sin(\pi x), \qquad \frac{\partial u}{\partial t}(x,0) = 0, \qquad u(0,t) = 0, \qquad u(10,t) = 0. \tag{15}$$

In this example the universe is the set $[0,10] \times [0,10]$. To see that the F-transform methodology is a good approximation for the exact solution of (6), we introduced random noise in the initial condition (15). We solved (6) using the standard centered FD-approximation for the wave equation. We applied this approximation in two cases, in the first case we solved the wave equation using this FD-approximation without noise, see figure 1a. In the second case, we solved the wave equation with random noise in the initial condition (15), see figure 1b. Then, we used the F-transform methodology to find the approximate solution of (6), see figure 1c.

5 Conclusion

In this work, we show that F-transform methodology can be used on FD schemes for a certain class of PDEs when the inital condition of such equations is noisy. The F-transform methodology gives a good result on noise reduction of the final FD solution, while classical FD methods are, in general, not designed to handle noise.

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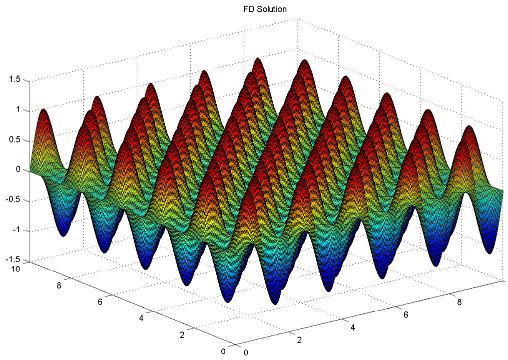
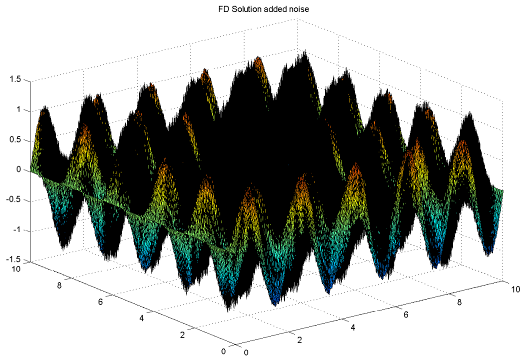
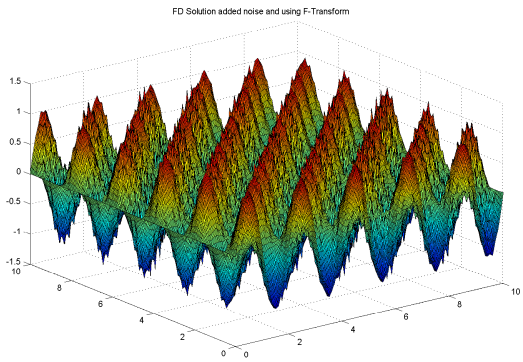


(a) Without noise.



(b) With noise.



(c) With noise and F-transform.

Figure 1: Finite-difference solution of equation (6)

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