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Continuous-Valued Octonionic Hopfield Neural Network

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In this paper, we generalize the famous Hopfield neural network to unit octonions. In the proposed model, referred to as the continuous-valued octonionic Hopfield neural network (CV-OHNN), the next state of a neuron is obtained by setting its octonionic activation potential to length one. We show that, like the traditional Hopfield network, a CV-OHNN operating in an asynchronous update mode always settles down to an equilibrium state under mild conditions on the octonionic synaptic weights.

Keywords. Octonions, Hopfield neural network, energy function, discrete-time dynamical system.

1 Introduction

The last few years witnessed an increasing interest on neural networks with values in multidimensional domains, such as complex-valued neural networks, quaternion-valued neural networks, octonion-valued neural networks, and networks based on Clifford algebras [7, 19]. One advantage of those networks, referred to as hypercomplex-valued neural networks, is that they treat multi-dimensional data as single entities [4,7,19]. For instance, complex-valued, quaternion-valued, and octonion-valued neural networks are able to process two, four, and eight dimensional data, respectively. Applications of hypercomplex-valued neural networks include control [1, 4], color image processing [14, 17, 18], and prediction [3, 16, 22].

In this paper, we present an octonion-valued generalization of the Hopfield network. The *Hopfield neural network* (HNN) is a single layer recurrent non-linear model which can be used for the storage and recall of vectors [6]. Apart from the storage and recall of vectors, the Hopfield network has been applied for solving optimization problems [8] and in computer vision modeling [20].

There exists a vast literature on hypercomplex versions of the Hopfield network which include generalizations using quaternions [9, 18], hyperbolic numbers [11, 12], Lie algebra [21], and Clifford algebra [24]. Recently, Kuroe and Iima introduced a class of octonionic Hopfield neural networks and provided conditions for the existence of an energy function for the stability analysis of their model [13]. On the downside, the stability of

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Table 1: Octonion Multiplication Table. The product between $\mathbf{i}_\kappa$ and $\mathbf{i}_\ell$ is situated in the intersection of the κ-th row and the ℓ-th column.

$\times$	$\mathbf{i_1}$	$\mathbf{i_2}$	$\mathbf{i_3}$	$\mathbf{i_4}$	$\mathbf{i_5}$	$\mathbf{i_6}$	$\mathbf{i_7}$
$\mathbf{i_1}$	-1	$\mathbf{i_4}$	$\mathbf{i_7}$	$-\mathbf{i_2}$	$\mathbf{i_6}$	$-\mathbf{i_5}$	$-\mathbf{i_3}$
$\mathbf{i_2}$	$-\mathbf{i_4}$	-1	$\mathbf{i_5}$	$\mathbf{i_1}$	$-\mathbf{i_3}$	$\mathbf{i_7}$	$-\mathbf{i_6}$
$\mathbf{i_3}$	$-\mathbf{i_7}$	$-\mathbf{i_5}$	-1	$\mathbf{i_6}$	$\mathbf{i_2}$	$-\mathbf{i_4}$	$\mathbf{i_1}$
$\mathbf{i_4}$	$\mathbf{i_2}$	$-\mathbf{i_1}$	$-\mathbf{i_6}$	-1	$\mathbf{i_7}$	$\mathbf{i_3}$	$-\mathbf{i_5}$
$\mathbf{i_5}$	$-\mathbf{i_6}$	$\mathbf{i_3}$	$-\mathbf{i_2}$	$-\mathbf{i_7}$	-1	$\mathbf{i_1}$	$\mathbf{i_4}$
$\mathbf{i_6}$	$\mathbf{i_5}$	$-\mathbf{i_7}$	$\mathbf{i_4}$	$-\mathbf{i_3}$	$-\mathbf{i_1}$	-1	$\mathbf{i_2}$
$\mathbf{i_7}$	$\mathbf{i_3}$	$\mathbf{i_6}$	$-\mathbf{i_1}$	$\mathbf{i_5}$	$-\mathbf{i_4}$	$-\mathbf{i_2}$	-1

the octonionic Hopfield networks of Kuroe and Iima is asserted by assuming that the activation function is, among other properties, injective. It turns out that the constraints imposed on the activation function may limit the applicability of the octonionic Hopfield network, for instance, as an associative memory [5].

In this paper, we introduce a time-discrete neural network that can be effectively used to implement associative memories. Specifically, we extend the *continuous-valued quaternionic Hopfield neural network* (CV-QHNN) to the octonion domain [23]. The resulting model, called *continuous-valued OHNN* (CV-OHNN), always settles down to an equilibrium state under the usual conditions.

This paper is organized as follows: Next section presents basic concepts on octonions. We introduce the CV-OHNN model and analyze its stability in Section 3. Section 4 presents some conclusions and provides perspectives of future works.

2 Basic concepts on octonions

Octonions, introduced by Graves in 1844, are 8-dimensional hyper-complex numbers that extend the complex and quaternion number system [2]. Apart from real, complex, and quaternion algebras, octonions form the unique normed and divisional algebra. Furthermore, although octonions are not as well known as the quaternions and complex numbers, they have applications in fields such as string theory, special relativity, and quantum logic [2].

An octonion x is a hypercomplex number which can be written in the form

$$x = x_0 + x_1\mathbf{i_1} + x_2\mathbf{i_2} + x_3\mathbf{i_3} + x_4\mathbf{i_4} + x_5\mathbf{i_5} + x_6\mathbf{i_6} + x_7\mathbf{i_7}, \tag{1}$$

where $x_0, \ldots, x_7$ are real numbers, and $\mathbf{i_1}, \ldots, \mathbf{i_7}$, typed here using boldface letters, are hyper-imaginary numbers that satisfy the multiplication rules given by Table 1. We denote by $\mathbb{O}$ the set of all octonions.

An octonion x can also be written as $x = x_0 + \vec{x}$, where x_0 and $\vec{x} = x_1\mathbf{i_1} + x_2\mathbf{i_2} + x_3\mathbf{i_3} + x_4\mathbf{i_4} + x_5\mathbf{i_5} + x_6\mathbf{i_6} + x_7\mathbf{i_7}$ are called, respectively, the real part and the vector part of x. The real and the vector part of an octonion x are also denoted by $\mathrm{Re}\{x\} := x_0$ and $\mathrm{Ve}\{x\} := \vec{x}$.

We would like to recall that the octonion algebra is neither commutative nor associative. Indeed, we have $\mathbf{i}_\kappa \mathbf{i}_\ell = -\mathbf{i}_\ell \mathbf{i}_\kappa$ for $\kappa \neq \ell$ and $(\mathbf{i}_\kappa \mathbf{i}_\ell)\mathbf{i}_\nu = -\mathbf{i}_\kappa(\mathbf{i}_\ell \mathbf{i}_\nu) \neq \mathbf{i}_\kappa(\mathbf{i}_\ell \mathbf{i}_\nu)$ for $\kappa \neq \ell \neq \nu$. Since the octonion algebra does not enjoy the associative property, it is not a Clifford algebra. Although the algebra of the octonions is not associative, the identity $\mathrm{Re}\{(xy)z\} = \mathrm{Re}\{x(yz)\}$ holds true for any $x, y, z \in \mathbb{O}$. This is an important property that we will use for the analyze of stability of the proposed CV-OHNN model.

The sum of two octonions is the octonion obtained by adding their corresponding components. The conjugate and the norm of an octonion x, denoted respectively by $\bar{x}$ and $|x|$, are defined by

$$\bar{x} = x_0 - \vec{x} \quad \text{and} \quad |x| = \sqrt{\bar{x}x} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2}. \tag{2}$$

We say that x is a *unit* octonion if $|x| = 1$. The set of all unit octonions is denoted by $\mathbb{S}_7$, i.e., $\mathbb{S}_7 = \{x \in \mathbb{O} : |x| = 1|\}$. Geometrically, $\mathbb{S}_7$ can be regarded as an hypersphere in $\mathbb{R}^8$.

3 Continuous-valued Octonionic Hopfield Neural Network

The neural network proposed by Hopfield (HNN) in 1982 have been designed to process n-bit vectors. Over the years, the Hopfield network has been generalized to work with models that treat multidimensional data as a single entity [9,10,13,15,23]. In this paper, we introduce the *continuous-valued octonionic Hopfield neural network* (CV-OHNN), which generalizes the Hopfield model to process vectors whose components are unit octonions. The novel CV-OHNNs can be used, for instance, to implement an associative memory designed for the storage and recall of vectors on $\mathbb{S}_7^n$. A CV-OHNN is defined as follows:

Definition 3.1 (Continuous-Valued Octonionic Hopfield Neural Network)**.** *Let $w_{ij} \in \mathbb{O}$, for $i, j = 1, \ldots, n$, denote the octonionic synaptic weight of the network. Given an input $\mathbf{x}(0) = [x_1, \ldots, x_n]^T \in \mathbb{S}_7^n$, we define recursively the sequence of octonion-valued vectors $\mathbf{x}(0), \mathbf{x}(1), \mathbf{x}(2), \ldots$ as follows*

$$x_i(t + \Delta t) = \begin{cases} \dfrac{v_i(t)}{|v_i(t)|}, & v_i(t) \neq 0 \\ x_i(t), & \text{otherwise}, \end{cases} \tag{3}$$

where

$$v_i(t) = \sum_{j=1}^{n} w_{ij} x_j(t), \tag{4}$$

is the activation potential of the ith neuron at iteration t.

Remark 3.1. *Throughout the paper, we assume that all the neurons of the CV-OHNN are updated asynchronously in one time unit. Thus, we have $\Delta t = 1/n$.*

Inspired by [13], we analyze the dynamics of the CV-OHNN model by means of the energy function

$$E(\mathbf{x}(t)) = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{Re}\left\{(\bar{x}_i(t)w_{ij})x_j(t)\right\}. \tag{5}$$

We know that the sequence $\{\mathbf{x}(t)\}_{t\geq 0}$ produced by the CV-OHNN is convergent if the strict inequality

$$\Delta E(t) = E(\mathbf{x}(t+\Delta t)) - E(\mathbf{x}(t)) < 0,$$

holds true whenever $\mathbf{x}(t+\Delta t) \neq \mathbf{x}(t)$. In this case, the time evolution of the CV-OHNN yields a minima of (5). In other words, the network settles down to a stationary state.

Theorem 3.1. *The sequence produced by* (3)*, in an asynchronous update mode, is convergent for any initial state* $\mathbf{x}(0) \in \mathbb{S}_7^n$ *if the synaptic weights satisfy* $w_{ij} = \bar{w}_{ji}$ *and* $w_{ii} \geq 0$ *for any* $i, j \in \{1, \ldots, n\}$*.*

Proof. First of all, from (5) we conclude that the energy function E is real-valued. Furthermore, it is not hard to shown that E is bounded from below.

Let us now show that E is strictly decreasing along any non-stationary trajectory. Since we adopted asynchronous update mode, we may suppose that only the kth neuron changed its state at iteration t, that is, we assume that $x_j(t+\Delta t) = x_j(t)$ for all $j \neq k$ and $x_k(t+\Delta t) \neq x_k(t)$. The energy function given by (5) evaluated at $\mathbf{x}(t)$ and $\mathbf{x}(t+\Delta t)$ satisfy respectively

$$\begin{aligned} E(\mathbf{x}(t)) = -\frac{1}{2}\Bigg(&\sum_{i\neq k}\sum_{j\neq k} \mathrm{Re}\left\{(\bar{x}_i(t)w_{ij})x_j(t)\right\} + \mathrm{Re}\left\{(\bar{x}_k(t)w_{kk})x_k(t)\right\} \\ &+ \sum_{j\neq k} \mathrm{Re}\left\{(\bar{x}_k(t)w_{kj})x_j(t)\right\} + \sum_{i\neq k} \mathrm{Re}\left\{(\bar{x}_i(t)w_{ik})x_k(t)\right\}\Bigg), \end{aligned}$$

and

$$\begin{aligned} &E(\mathbf{x}(t+\Delta t)) = \\ &-\frac{1}{2}\Bigg(\sum_{i\neq k}\sum_{j\neq k} \mathrm{Re}\left\{(\bar{x}_i(t+\Delta t)w_{ij})x_j(t+\Delta t)\right\} + \mathrm{Re}\left\{(\bar{x}_k(t+\Delta t)w_{kk})x_k(t+\Delta t)\right\} \\ &+ \sum_{j\neq k} \mathrm{Re}\left\{(\bar{x}_k(t+\Delta t)w_{kj})x_j(t+\Delta t)\right\} + \sum_{i\neq k} \mathrm{Re}\left\{(\bar{x}_i(t+\Delta t)w_{ik})x_k(t+\Delta t)\right\}\Bigg). \end{aligned}$$

By hypothesis, $w_{kk} = \bar{w}_{kk}$. Thus, w_{kk} is a real number. Moreover, we know that $x_k(t)$ and $x_k(t+\Delta t)$ are unit octonions, i.e, $|x_k(t)| = |x_k(t+\Delta t)| = 1$. Therefore, we have

$$\mathrm{Re}\left\{(\bar{x}_k(t)w_{kk})x_k(t)\right\} = \mathrm{Re}\left\{(w_{kk}\bar{x}_k(t))x_k(t)\right\} = w_{kk}\mathrm{Re}\left\{\bar{x}_k(t)x_k(t)\right\} = w_{kk}|x_k(t)|^2 = w_{kk}.$$

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Similarly, the identity $\mathrm{Re}\left\{(\bar{x}_k(t+\Delta t)w_{kk})x_k(t+\Delta t)\right\} = w_{kk}$ holds true. Therefore, the variation of the energy at time t satisfies

$$\begin{aligned}\Delta E &= E(\mathbf{x}(t+\Delta t)) - E(\mathbf{x}(t)) \\ &= -\frac{1}{2}\left[\sum_{j\neq k}(\bar{x}_k(t+\Delta t)-\bar{x}_k(t))(w_{kj}x_j(t)) + \sum_{i\neq k}(\bar{x}_i(t)w_{ik})(x_k(t+\Delta t)-x_k(t))\right].\end{aligned}$$

From the equality $w_{ij} = \bar{w}_{ji}$, we conclude that the conjugate of $\sum_{i\neq k}\bar{x}_i(t)w_{ik}(x_k(t+\Delta t) - x_k(t))$ is $\sum_{j\neq k}(\bar{x}_k(t+\Delta t)-\bar{x}_k(t))w_{kj}x_j(t)$. In addition, we are able to express ΔE by means of the equation

$$\begin{aligned}\Delta E &= -\,\mathrm{Re}\left\{(\bar{x}_k(t+\Delta t)-\bar{x}_k(t))\sum_{j\neq k}w_{kj}x_j(t)\right\} \\ &= -\mathrm{Re}\left\{(\bar{x}_k(t+1)-\bar{x}_k(t))(v_k(t)-w_{kk}x_k(t))\right\},\end{aligned}$$

where $v_k(t) = \sum_{j=1}^{n} w_{kj}x_j(t)$ is the activation potential of the kth neuron at iteration t.

As we are considering $x_k(t+1) \neq x_k(t)$, we must have $v_k(t) \neq 0$. Moreover, we have from (3) that $v_k(t) = Ax_k(t+1)$, where $A = |v_k(t)| > 0$. Thus, we conclude that

$$\begin{aligned}\Delta E &= -\,\mathrm{Re}\left\{(\bar{x}_k(t+\Delta t)-\bar{x}_k(t))(Ax_k(t+\Delta t)-w_{kk}x_k(t))\right\} \\ &= -\,\mathrm{Re}\left\{A(\bar{x}_k(t+\Delta t)-\bar{x}_k(t))x_k(t+\Delta t) + w_{kk}(\bar{x}_k(t)-\bar{x}_k(t+\Delta t))x_k(t)\right\} \\ &= -\left[A\mathrm{Re}\left\{1-\bar{x}_k(t)x_k(t+\Delta t)\right\} + w_{kk}\mathrm{Re}\left\{1-\bar{x}_k(t+\Delta t)x_k(t)\right\}\right] \\ &= -\,(A+w_{kk})\left(1-\mathrm{Re}\left\{\bar{x}_k(t)x_k(t+\Delta t)\right\}\right).\end{aligned}$$

Now, from Cauchy-Schwarz inequality, we derive

$$\begin{aligned}&\mathrm{Re}\left\{\bar{x}_k(t)x_k(t+\Delta t)\right\} = x_{k0}(t)x_{k0}(t+\Delta t) + x_{k1}(t)x_{k1}(t+\Delta t) + x_{k2}(t)x_{k2}(t+\Delta t) \\ &+ x_{k3}(t)x_{k3}(t+\Delta t) + x_{k4}(t)x_{k4}(t+\Delta t) + x_{k5}(t)x_{k5}(t+\Delta t) + x_{k6}(t)x_{k6}(t+\Delta t) \\ &+ x_{k7}(t)x_{k7}(t+\Delta t) < |x_k(t)||x_k(t+\Delta t)| = 1,\end{aligned}$$

because $x_k(t+1)$ and $x_k(t)$ are *not parallel vectors* in $\mathbb{R}^8$. Therefore, the inequality $1-\mathrm{Re}\left\{\bar{x}_k(t)x_k(t+\Delta t)\right\} > 0$ holds. Since $A > 0$ and $w_{kk} \geq 0$, $A + w_{kk}$ is also positive. Consequently, we have $\Delta E < 0$ if $x_k(t+\Delta t) \neq x_k(t)$ for some index $k \in \{1, 2, \ldots, n\}$. Hence, the network in asynchronous update mode settles down to a stationary state. □

4 Concluding Remarks

In this paper we introduced the *continuous-valued octonionic Hopfield neural network* (CV-OHNN). In few words, the CV-OHNN is based on the activation function that normalizes the potential of activation to length one. As a consequence, this octonionic network

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can be implemented and analyzed easily. Indeed, we presented an energy function for the CV-OHNN based on [13]. Also, we showed that this real-valued bounded function is strictly decreasing along any non-stationary trajectory. Therefore, the network using asynchronous update always settles down to an equilibrium state.

In the future, we plan to investigate the performance of this model as an associative memory. In particular, we also plan to study its storage capacity and noise tolerance.

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