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The Linear Kawahara Equation on the Half-Line

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Abstract. We study the stabilization of global solutions of the Linear Kawahara (K) equation posed on the right half-line, under the effect of a localized damping mechanism. In this work we analyze the existence, uniqueness and regularity of solutions for the (K) equation, using semigroups theory and since this system is defined on an unbounded domain, a special multiplier argument is showed. To prove the exponential decay of the energy associated to (K) system, due to a lack of compactness, we use local compactness arguments and multipliers techniques. The Kawahara equation describes the evolution of small amplitude long waves in problems arising in fluid dynamics.

Keywords. exponential decay; stabilization; Kawahara equation; half-line domain.

1 Introduction

We consider the Kawahara linear system on the half-line under the presence of a localized damping

$$\left|\begin{array}{ll} u_t + u_x + u_{xxx} + \eta u_{xxxxx} + a(x)u = 0 & \text{in} \quad \mathbb{R}^+ \times (0,+\infty) \\ u(0,t) = u_x(0,t) = 0 & \text{for all} \quad t \geq 0 \\ u(x,0) = u_0(x) & \text{in} \quad \mathbb{R}^+. \end{array}\right. \tag{1.1}$$

Here $a = a(x)$ is a non-negative function belonging to $L^\infty(\mathbb{R}^+)$ and moreover, we will assume that $a(x) \geq a_0 > 0$ a.e. in an open non-empty subset ω of $\mathbb{R}^+$, where the damping is acting effectively. The constant η is a negative real number.
It is well known that the above system, in absence of damping, describes the one-dimensional evolution of small amplitude long waves in several problems arising in fluid dynamics.
In this model the conservative dispersive effect is represented by the term $(u_{xxx}+\eta u_{xxxxx})$.
The Kawahara equation is given by

$$u_t + uu_x + \alpha u_{xxx} + \beta u_{xxxxx} = 0 \tag{1.2}$$

where α and β are constants representing the effect of dispersion. This equation is a model for plasma wave, capilarity-gravity water waves and other dispersive phenomena when the

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cubic KdV-type is weak. Kawahara [4] pointed out that it happens when the coefficient of the third order derivative in the KdV equation becomes very small or even zero. It is necessary to take account of the higher order effect of dispersion in order to balance the nonlinear effect. Kakutani and Ono [3] showed that for a critical value of angle between the magneto-acoustic wave in a cold collision-free plasma and the external magnetic field, the third order derivative term in the KdV equation vanishes and may be replaced by the fifth order derivative term. Following this idea, Kawahara [4] studied a generalized nonlinear dispersive equation which has a form of the KdV equation with an additional fifth order derivative term.

Dispersive problems have been object of intensive research, notably in exact controllability and the stabilization of the system. About the Kawahara system in bounded domain, we can consider, for instance, the following papers: [13, 15, 16] and references therein. In the case with periodic boundary conditions, see [14].

Works involving the KdV system in bounded domain can be cited, such as [7] and [8,11,12]. In unbounded domain we have, for instance, the following articles: [6, 9].

In this work we study the problem of the decay of the energy associated to system (1.1) as $t \to +\infty$ in the presence of the localized damping term $a(x)u$.

We denote by $||f||$ the $L^2(\mathbb{R}^+)$-norm of the function f.

The total energy associated with the system (1.1) is defined by

$$E(t) = \frac{1}{2}\int_0^{+\infty} |u(x,t)|^2\, dx \, = \frac{1}{2}||u(t)||^2$$

Using the above boundary conditions we prove that

$$\frac{dE}{dt} = \frac{\eta}{2}|u_{xx}(0,t)|^2 - \int_0^{+\infty} a(x)|u(x,t)|^2\, dx \leq 0\,, \forall t > 0.$$

So, $E(t)$ is a nonincreasing function of time.

As expected this work is devoted to analyze the following questions: Does the energy $E(t) \to 0$ as $t \to +\infty$? Is it possible to find a rate of decay of the energy?

The natural idea to answer the above questions is to follow closely previous articles on Kawahara system as [15, 16].

However, the present system is defined on the half-line and therefore others difficulties arise as the regularity of solution and the lack of compactness.

In Section 2, we consider the initial data belonging to $L^2(\mathbb{R}^+)$ then, using semigroup theory and multipliers techniques we prove the existence and uniqueness of global solutions for the system (1.1) in $C([0, +\infty); L^2(\mathbb{R}^+))$.

In the section 3, inspired in the article of Kruzhkov and Faminskii [5] (see also Faminskii [2]) for KdV system, we use multipliers techniques and some special regular function defined in half-line to prove that the solution of system (1.1) belongs to $L^2_{loc}(0, +\infty; H^2_0(\mathbb{R}^+))$

In section 4, we prove the exponential decay of the energy associated to system (1.1) using local compactness results.

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2 Existence and uniqueness

In this section we study the global existence and uniqueness of the solutions for the following problem:

$$\left| \begin{array}{l} u_t + u_x + u_{xxx} + \eta u_{xxxxx} = 0 \qquad \text{in} \quad \mathbb{R}^+ \times (0,+\infty). \\ u(0,t) = 0 = u_x(0,t) \qquad \text{for all} \quad t > 0. \\ u(x,0) = u_0(x) \qquad \text{in} \quad \mathbb{R}^+, \end{array} \right. \tag{2.1}$$

where η is a negative real number and u_0 belongs to $L^2(\mathbb{R}^+)$.

Theorem 2.1. (Existence and uniqueness)
If u_0 belongs to $L^2(\mathbb{R}^+)$ and $\eta < 0$, then, the problem (2.1) has a unique solution u belonging to $C([0,+\infty); L^2(\mathbb{R}^+))$ and moreover

i) $||u||_{C([0,+\infty);L^2(\mathbb{R}^+))} \leq ||u_0||$

ii) $u_{xx}(0,\cdot)$ *belongs to* $L^2(0,+\infty)$ *and* $||u_{xx}(0,\cdot)||^2_{L^2(0,+\infty)} \leq -\dfrac{1}{\eta}||u_0||^2$.

The energy dissipation law,

$$\frac{1}{2}\frac{d}{dt}||u(t)||^2 = \frac{\eta}{2}|u_{xx}(0,t)|^2, \forall t > 0, \tag{2.2}$$

holds.

Proof: Let A denote the linear operator $Av = -v' - v''' - \eta v'''''$ defined on the dense domain $D(A) \subset L^2(\mathbb{R}^+)$, where $D(A) = \{v \in H^5(0,L) : v(0) = v'(0) = 0\}$.
Let $v \in D(A)$, then, using integration by parts and definition of $D(A)$, we have:

$$(v, Av) = \int_0^{+\infty} v(-v' - v''' - \eta v''''')\, dx = \frac{\eta}{2}|v''(0)|^2$$

Since $\eta < 0$, we can show that the linear operator A is dissipative.
Now, we see that the adjoint operator A^* is defined by $A^*w = w' + w''' + \eta w'''''$, where w belongs to $D(A^*) = \{w \in H^5(0,L) : w(0) = w'(0) = w''(0) = 0\}$.
So, using integration by parts again and definition of $D(A^*)$, we obtain:

$$(w, A^*w) = \int_0^{+\infty} w(w' + w''' + \eta w''''')\, dx = 0.$$

We can see that the adjoint operator A^* is also dissipative and a closed operator.
On the other hand, using integration by parts, we can prove that $D(A^{**}) = D(A)$ and $A^{**} = A$, hence it follows that A is closed operator. Therefore, from classical results in semigroup theory (Pazy [10] cor. 4.4, chapter 1), we know that A generates a strongly continuous semigroup of contractions in $L^2(\mathbb{R}^+)$.

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Let $\{S(t)\}_{t\geq 0}$ be the semigroup of contractions generated by A. So, the unique mild solution $u(t) = S(t)u_0, t \geq 0$ belongs to $C([0,+\infty); L^2(\mathbb{R}^+))$ and it satisfies:
$||u||_{C([0,+\infty);L^2(\mathbb{R}^+))} \leq ||u_0||$.
The hidden regularity in ii) and the energy dissipation law follow by multipliers techniques and boundary conditions. ■

3 Regularity

In this section we prove some regularity for the solution of the problem (2.1), that is, we show that the solution u belongs to $L^2_{loc}((0,+\infty); H^2_0(\mathbb{R}^+))$.
Inspired in the work of the Kruzhkov and Faminskii [5] for KdV type equation (see also Faminskii [2]), we use multipliers techniques and a particular result about smooth functions to prove this result. At first, we consider an important lemma:

Lemma 3.1. *If u belongs to $H^2(\mathbb{R}^+)$ then there exists a constant $K > 0$ such that:*

$$\int_0^\infty u_x^2 dx \leq K \left(\epsilon \int_0^\infty u_{xx}^2 dx + \epsilon^{-1} \int_0^\infty u^2 dx \right), \ \ \forall \epsilon > 0 \tag{3.1}$$

and K is independent of ϵ.

This Lemma is a particular case of Lemma 5.5 (see also the Remark 5.7) in Adams and Fournier [1].
Now, we can consider the following proposition:

Proposition 3.1. *Let u be a solution of system (2.1). Then, for any $x_0 \in \mathbb{R}^+$*

$$\int_0^T \int_{x_0}^{x_0+1} u_x^2 \, dxdt + \int_0^T \int_{x_0}^{x_0+1} u_{xx}^2 \, dxdt \ \leq \ c(\eta, \psi_0, K, T)\|u_0\|^2. \tag{3.2}$$

Proof: We begin considering $u_0 \in D(A)$. In this case, by classical semigroup theory, we know that the solution u belongs to $C^1((0\,,+\infty)\,; D(A))$ and moreover, by Theorem 2.1, we have: $||u||_{C([0,+\infty);L^2(\mathbb{R}^+))} \leq ||u_0||$.
Now, we define a special nondecreasing function $\psi_0 \in C^\infty(\mathbb{R}^+)$. So, fixing $x_0 \in \mathbb{R}^+$, multiplying the equation in (2.1) by $u(x,t)\psi_0(x - x_0 + \frac{1}{2})$, integrating by parts over $(0,T) \times \mathbb{R}^+$ and using the Lemma 3.1 we obtain (3.2).

Theorem 3.1. (Regularity)
*If u is a solution of the problem (2.1), then u belongs to $L^2_{loc}((0,+\infty); H^2_0(\mathbb{R}^+))$.
Moreover, $\|u\|_{L^2_{loc}((0,+\infty);H^2_0(\mathbb{R}^+))} \leq c\,\|u_0\|$.*

This Theorem is proved using the above Proposition and multipliers techniques.

4 Exponential Decay of the Energy

In this section, we analyze the system (1.1), where the assumptions about the damping a were presented in the Introduction.
We observe that the system is also well posed in $L^2(\mathbb{R}^+)$. This can be easily proved considering the problem (1.1) as a perturbation of the case $a \equiv 0$, that is, the problem (2.1).
We begin considering a result of observability:

Proposition 4.1. (observability)
Under the assumptions of Theorem 2.1, for any $T > 0$, we have:

$$\frac{1}{2}||u_0||^2 \leq \frac{1}{2T}\int_0^T \int_0^{+\infty} |u(x,t)|^2\, dxdt - \frac{\eta}{2}\int_0^T |u_{xx}(0,t)|^2\, dt + \int_0^T \int_0^{+\infty} a(x)|u(x,t)|^2\, dxdt. \tag{4.3}$$

for all solutions of (1.1).

To prove this Proposition it is sufficient to multiply the equation in (1.1) by $(T-t)u$, to integrate by parts over $(0,T)\times(0,\infty)$ and to use boundary conditions.

Remark 4.1. *We observe that if we multiply the equation in (1.1) by u, integrate by parts over $(0,T)\times(0,\infty)$ and use boundary conditions, we have:*

$$-\frac{\eta}{2}\int_0^T |u_{xx}(0,t)|^2\, dt + \int_0^T \int_0^{+\infty} a(x)|u(x,t)|^2\, dxdt \leq \frac{1}{2}||u_0||^2 \tag{4.4}$$

Theorem 4.1. (Stabilization)
Assume $a = a(x)$ is a non-negative function belonging to $L^\infty(\mathbb{R}^+)$ and moreover $a \geq a_0 > 0$ a.e. in an open non-empty subset ω of $\mathbb{R}^+$. Then, there exist $c > 0$ and $\mu > 0$ such that

$$E(t) \leq c||u_0||^2\, e^{-\mu t} \tag{4.5}$$

for all $t \geq 0$ and all solution of (1.1) with $u_0 \in L^2(\mathbb{R}^+)$.

By the Proposition 3.1 and Remark 4.1, it is sufficient to prove

$$\frac{1}{2}\int_0^T \int_0^{+\infty} |u(x,t)|^2\, dxdt \leq c_1 \left\{ \frac{-\eta}{2}\int_0^T |u_{xx}(0,t)|^2\, dt + \int_0^T \int_0^{+\infty} a(x)|u(x,t)|^2\, dxdt \right\} \tag{4.6}$$

for some positive constant c_1, independent of the solution u. In fact, the decay of the Energy follows by semigroup property.
To prove (4.6) we argue by contradiction, following a "compactness-uniqueness" argument (see, for instance, Zuazua [17]).

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