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Pólya's Looking Back: an Example with Leibniz's Integrating Factor and Second Order Linear Differential Equations

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Abstract. A teacher who wants to present ordinary differential equation subjects with Pólya's flavour has had a hard time on finding good examples which fits that aim. It might be due the fact that many well-done solutions obtained by using Pólya's approach, for some reason, were never even submitted for publishing by its authors. Here we describe how looking back at Leibniz's integrating factor lead us to an integrating factor method for non-homogeneous second order linear equations. This approach avoids the need of knowledge on complex number's topics such as Euler's identity and complex exponential.

Keywords. Polya's Looking Back, Problem Solving, Differential Equation, Leibniz's Integrating Factor

1 Introduction

Most introductory books on ordinary differential equations – for instance, [1] – present the integrating factor method for solving first order linear differential equations

$$y'(t) + p(t)\, y(t) = q(t)\,, \tag{1}$$

where p and g are given continuous functions. We owe this method to Leibniz and its first step involves multiplying equation (1) by a certain function $\mu(t)$, thus

$$\mu(t)\, y'(t) + p(t)\, \mu(t)\, y(t) = \mu(t)\, q(t)\,. \tag{2}$$

The second step is to pick a function $\mu(t)$ such that $\mu'(t) = p(t)\, \mu(t)$, for instance, $\mu(t) = e^{\int_{t_0}^{t} p(\tau)\, d\tau}$. Therefore, equation (2) becomes

$$(\mu(t)\, y(t))' = \mu(t)\, g(t)\,. \tag{3}$$

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Then, by integration we obtain

$$\mu(t)\, y(t) = \int_{t_0}^{t} \mu(\tau)\, g(\tau) d\tau + k\,,$$

from which it follows that the general solution of Equation (1) is

$$y(t) = \frac{1}{\mu(t)} \left(\int_{t_0}^{t} \mu(\tau)\, g(\tau) d\tau + k \right)\,, \tag{4}$$

where k is an arbitrary constant.

On his best-selling [3], Pólya separates the problem solving process into four phases, namely, *understanding the problem*, *devising a plan*, *carrying out the plan* and *looking back*. The influence of Polya's four phases on mathematical education can be noted, for instance, by the fact that almost every article submitted to the 1980 yearbook of the National Council of Teachers of Mathematics, which were concerned on problem-solving, references to them [2]. However, a teacher who wants to present ordinary differential equation subjects with Pólya's flavour has had a hard time on finding good examples which fits that aim. It might be due the fact that many well-done solutions obtained by using Pólya's approach, for some reason, were never even submitted for publishing by its authors.

This article focus on the last phase, the looking back, whose suggestions are: *can you check the result? can you check the argument?*; *can you derive the result differently? can you see it at a glance?*; *can we use the result, or the method, for some other problem?*. Notice that one could easily check that y given by (4) is a solution of equation (1). Furthermore, it is not hard to check the arguments in each step of Leibniz's integrating factor method too. There is another classical way for solving equation (1): the variation of parameters method. However, can we use the Leibniz's integrating factor method for some other problem? More specifically, let b and c be two given constants and consider the second order linear differential equation

$$y''(t) + b\, y'(t) + c\, y(t) = g(t)\,. \tag{5}$$

Question 1.1. *Can we use a sort of Leibniz's integrating factor method for equation (5)?*

Our main purpose was to describe, as good as we could, how looking back at Leibniz's integrating factor lead us to an integrating factor method for equation (5). Although the authors could not find the solution presented here elsewhere, they are not quite sure that it is a new one. The next section describes how the first partial answer to the question 1.1 came out: the identical roots case. That first achievement gave us hope to persist on our task and eventually we have surpassed the distinct real roots case as reported in Section 3. Once the first and second rounds were gone, we were confident enough to the last one, the complex roots case, presented in Section 4.

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2 Identical Roots

In the beginning Leibniz solved equation (3) and we saw his solution, that it was good. And then, we consider the equation

$$(\mu(t)\, y(t))'' = \mu(t)\, g(t)\,, \tag{6}$$

which is similar to equation (3) and that could be easily solved too. Equation (6) can be rewritten as

$$\mu(t)\, y''(t) + 2\mu'(t)y'(t) + \mu''(t)y(t) = \mu(t)\, g(t)\,, \tag{7}$$

then, once you have seen the Leibniz integrating factor method, you knows that if

$$\begin{cases} \mu'(t) = (b/2)\, \mu(t)\,, \\ \mu''(t) = c\, \mu(t)\,, \end{cases} \tag{8}$$

it is all settled and done.

The general solution of the first equation in (8) is given by $\mu(t) = k\, e^{\frac{b}{2}\, t}$. Thus, substituting $\mu(t) = k\, e^{\frac{b}{2}\, t}$ into the second equation of (8), we have $(b^2/4)\, k\, e^{\frac{b}{2}\, t} = c\, k\, e^{\frac{b}{2}\, t}$. Hence, whenever we have $b^2 - 4c = 0$, the integrating factor $\mu(t) = e^{\frac{b}{2}\, t}$ transforms our original equation (5) into

$$\left(e^{\frac{b}{2}\, t} y(t)\right)'' = e^{\frac{b}{2}\, t} g(t)\,.$$

Thus

$$(e^{\frac{b}{2}\, t} y(t))' = \int_{t_0}^{t} e^{\frac{b}{2}\, \tau} g(\tau)\, d\tau + c_1\,. \tag{9}$$

If we denote

$$h(t) = \int_{t_0}^{t} e^{\frac{b}{2}\, \tau} g(t)\, d\tau\,, \tag{10}$$

then by integration of equation (9) we obtain

$$y(t) = e^{-\frac{b}{2}\, t} \left(\int_{t_0}^{t} h(\tau)\, d\tau + c_1 t + c_2 \right)\,. \tag{11}$$

In particular, we can find the general solution of exercise 7 at page 189 of [1]:

Example 2.1. *Solve the differential equation*

$$y''(t) + 4y'(t) + 4y(t) = t^{-2} e^{-2t}\,, \quad t > 0\,. \tag{12}$$

Solution: Since $b = 4$ and $c = 4$, the discriminant is zero. Therefore, the integrating factor can be $\mu(t) = e^{2t}$. Moreover, from equation (10), we have

$$h(t) = \int_{t_0}^{t} e^{2\tau} \tau^{-2} e^{-2\tau} d\tau = \int_{t_0}^{t} \tau^{-2} d\tau = -t^{-1} + c_1\,, \tag{13}$$

Thus, from equations (11) and (13), we find that the general solution of (12) is given by

$$y(t) = e^{-2t} \left(-\int_{t_0}^{t} \tau^{-1} d\tau + c_1\, t + c_2 \right) = -e^{-2t} \ln(t) + c_1 t e^{-2t} + c_2 e^{-2t}\,.$$

3 Distinct Real Roots

In order to employ de integrating factor idea, our left side of Equation (5) shall be like

$$\mu(t)\, y''(t) + \nu(t)\, y'(t) + \phi(t)\, y(t)\,.$$

Moreover, this left side must be in someway reduced to a problem that we know how do solve. After some period of time of thinking and resting we realize that if our left side could be reduced to

$$\big(\mu(t)\ \left[y'(t) + k\, y(t)\right]\big)'\ ,$$

the question could be aswered. Notice that

$$\big(\mu(t)\,[y'(t) + k\, y(t)]\big)' = \mu(t)y''(t) + [\mu'(t) + k\,\mu(t)]\, y'(t) + k\,\mu'(t)\, y(t) = q(t) \tag{14}$$

Therefore, the left side of our original Equation (5) will become (14) if

$$\begin{cases} \mu'(t) + k\,\mu(t) = b\,\mu(t)\,, \\ \mu'(t) = (c/k)\,\mu(t)\,. \end{cases} \tag{15}$$

If we choose $k \neq b$ and $k \neq 0$, then, solving the system (15), we may have

$$\mu(t) = e^{(b-k)t} = e^{\frac{c}{k}\,t}\,. \tag{16}$$

Thus, it is sufficient to make $b - k = (c/k)$, and hence we can choose

$$k = \frac{b + \sqrt{b^2 - 4c}}{2}\,. \tag{17}$$

From Equations (16) and (17), we can pick as our integrating factor $\mu(t) = e^{\alpha t}$, where

$$\alpha = \frac{b - \sqrt{b^2 - 4c}}{2}\,. \tag{18}$$

Now our problem can be reduced to

$$\big(e^{\alpha\, t}\ \left[y'(t) + k\, y(t)\right]\big)' = e^{\alpha\, t} g(t)\,. \tag{19}$$

If we denote

$$h(t) = \int_{t_0}^{t} e^{\alpha\,\tau} g(\tau)\, d\tau\,, \tag{20}$$

then,

$$y'(t) + k\, y(t) = h(t)\, e^{-\alpha\, t} + c_1\, e^{-\alpha\, t}\,. \tag{21}$$

At last, from the integrating factor for first order linear differential equations, we obtain

$$y(t) = e^{-k\,t} \left(\int_{t_0}^{t} h(\tau)\, e^{(k-\alpha)\,\tau}\, d\tau + c_1 \int_{t_0}^{t} e^{(k-\alpha)\,\tau}\, d\tau + c_2 \right) \tag{22}$$

For instance, consider the exercise 1 at page 183 of [1]:

Example 3.1. *Solve the differential equation*

$$y''(t) - 2\,y'(t) - 3\,y(t) = 3\,e^{2t}\,.$$

Solution: Since $b = -2$ and $c = -3$, we have

$$k = \frac{2 + \sqrt{2^2 - 4.(-3)}}{2} = 1 \quad \text{and} \quad \alpha = \frac{2 - \sqrt{2^2 - 4(-3)}}{2} = -3\,.$$

Moreover, from equation (20),

$$h(t) = \int_{t_0}^{t} 3\,e^{2\,\tau}\,e^{-3\,\tau}\,d\tau = -3\,e^{-t} + c\,. \tag{23}$$

From equation (22),

$$y(t) = e^{-t}\left(\int_{t_0}^{t}\left(-3\,e^{3\,\tau} + c\,e^{4\,\tau}\right)\,d\tau + c_1\int_{t_0}^{t} e^{4\,\tau}\,d\tau + c_2\right) = -e^{2\,t} + k_1\,e^{3\,t} + k_2\,e^{-t}\,.$$

4 Complex Roots

Instead of the general problem, let us consider the following example:

Example 4.1. *Find the solution of the differential equation*

$$y''(t) + 2\,y'(t) + 4\,y(t) = 0\,. \tag{24}$$

Since $b = 2$ and $c = 4$ – *i.e.*, the discriminant is negative – example 4.1 can not be solved in those ways presented in the previous sections. After deal with those two previous cases, it was, in some sense, natural for us reduce our original problem to an equation like

$$\big(\mu(t)\,[y'(t) + \nu(t)\,y(t)]\big)' = \mu(t)\,y''(t) + [\mu'(t) + \nu(t)\mu(t)]\,y'(t) + [\mu(t)\nu(t)]'\,y(t) = 0\,. \tag{25}$$

Notice that it differs from equation (14): we have now $\nu(t)$ instead of a given constant k. Our hope was that the flexibility of choosing a function $\nu(t)$ could gave us the answer. Multiplying equation (24) by $\mu(t)$, we have

$$\mu(t)y''(t) + 2\mu(t)y'(t) + 4\mu(t)y(t) = 0\,. \tag{26}$$

Hence, equation (26) would be reduced to equation (25), whenever

$$\begin{cases} \mu'(t) + \nu(t)\,\mu(t) = 2\,\mu(t)\,, \\ \mu'(t)\,v(t) + \mu(t)\,v'(t) = 4\mu(t)\,. \end{cases} \tag{27}$$

The first equation of system (27) can be rewritten as $\mu'(t) = (2 - \nu(t))\mu(t)$ for which

$$\mu(t) = e^{2t - \int_{t_0}^{t} v(\tau)\,d\tau} \tag{28}$$

is a solution. Furthermore, substituting $\mu'(t) = (2 - \nu(t))\mu(t)$ in the second equation of system (27), we have

$$(2 - \nu(t))\mu(t)\,\nu(t) + \mu(t)\,\nu'(t) = 4\mu(t)\,,$$

or equivalently,

$$(2 - \nu(t))\,\nu(t) + \nu'(t) = 4\,. \tag{29}$$

Equation (29) is a separable equation. Indeed, we have

$$\frac{1}{\nu^2 - 2\nu + 4}\,d\nu = dt \quad \therefore \quad \frac{1}{(\nu-1)^2 + 3}\,d\nu = dt\,.$$

Thus

$$\frac{1}{\sqrt{3}}\arctan\left(\frac{v-1}{\sqrt{3}}\right) = t + c$$

Setting $c = 0$, we have $v(t) = 1 + \sqrt{3}\,\tan(\sqrt{3}\,t)$ from which follows that

$$\int_{t_0}^{t} v(\tau)d\tau = t + \ln|\sec(\sqrt{3}t)| + c\,. \tag{30}$$

Therefore, from Equations (28) and (30), we can pick $\mu(t) = e^t\cos(\sqrt{3}t)$ as our integrating factor. Thus

$$e^t\,\cos(\sqrt{3}\,t)\,\Big[y'(t) + \Big[1 + \sqrt{3}\,\tan(\sqrt{3}\,t\Big]\,y(t)\Big] = c_1\,,$$

or equivalently,

$$y'(t) + \Big[1 + \sqrt{3}\,\tan(\sqrt{3}\,t)\Big]\,y(t) = c_1\,e^{-t}\,\sec(\sqrt{3}\,t)\,. \tag{31}$$

The integrating factor for Equation (31) can be $\mu_1(t) = e^t\,\sec(\sqrt{3}t)$. Hence, we have

$$\Big(e^t\sec(\sqrt{3}\,t)\,y(t)\Big)' = c_1\sec^2(\sqrt{3}\,t)\,.$$

Thus, by integration, we obtain

$$e^t\sec(\sqrt{3}\,t)\,y(t) = \int_{t_0}^{t} c_1\sec^2(\sqrt{3}\,t)\,d\tau = c_1\frac{1}{\sqrt{3}}\tan(\sqrt{3}\,t) + c_2\,.$$

Finally,

$$y(t) = e^{-t}\,\Big(a_1\sin(\sqrt{3}\,t) + c_2\cos(\sqrt{3}\,t)\Big)\,.$$

And now the reader is ready for the following two Exercises:

Exercise 4.1. *Solve by the integrating factor method the problem*

$$y''(t) + 2y'(t) + 5y(t) = 3\sin(2t)\,.$$

Exercise 4.2. *Solve by the integrating factor method the problem*

$$y''(t) + b\,y'(t) + c\,y(t) = g(t),$$

where that b and c are given real numbers for which $b^2 - 4c < 0$.

The solution of exercise 4.2 completes the answer of question 1.1.

5 Conclusion

The reader may notice that

- this approach deals directly with non-homogeneous second order linear equations;
- it also avoids the need of knowledge about some complex number topics such as complex exponentiation and Euler's identity;
- on other hand, the linear algebra character of the homogenous solution plus a particular solution do not emerge here as naturally as in the usual approaches for second order linear differential equations.

The presentation of this answer for the question 1.1, for instance, may last about three hours. Therefore, it have been difficult for us to both cover all the classical contents of ordinary differential equations and provide a good number examples *a la* Polya. As an alternative, we are creating and typing several Polya's problem solving examples and providing to our students as a complementary material.

Polya's looking back phase is always open-ended, any solved solution is the starting point for another looking back phase. For instance, once you saw how we have answered question 1.1 into three steps,

Exercise 5.1. *Can you answer question 1.1 at a glance?*

Moreover, in [4], the Leibniz's integrating factor idea is adapt in order to solve first order linear difference equations.

Exercise 5.2. *Can you adapt this method for solving second order linear difference equations?*

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