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An Introduction to Projection Fuzzy Associative Memories in Complete Inf-Semilattices

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Abstract. In this paper we introduced a projection fuzzy associative memory (PFAM) based on complete inf-semilattice (cisl) derived from complete lattice $[0,1]^n$ by using reference function is called a semilattice based projection fuzzy associative memory (SL-PFAM). Since the PFAM project to input pattern into the set of all max-C (C means conjunction) combination of the fundamental memories set. In experimental section, we compare the performance of SL-PFAM using fuzzy morphological associative memories (FMAMs) as reference functions for the reconstruction of gray scale images that were incomplete patterns and corrupted by different types of noise.

Keywords. fuzzy associative memory, mathematical morphology, complete inf-semilattice, reference element, projection fuzzy associative memory, image restoration.

1 Introduction

Generally speaking, *associative memories* (AMs) allow for the storage of pattern associations and the retrieval of the desired output pattern on presentation of a possibly corrupted or incomplete version of an input pattern. Mathematically speaking, AMs is *content-addressable structure* that maps a set of input patterns to a set of output patterns. AMs can be either *hetero-associative or auto-associative*, where the input and output vectors range over *different vector spaces* or over the *same vector space*. If an associative memory provide a fuzzy rule base to be stored that is the inputs are the *degrees of membership*, and the outputs are the fuzzy system's output. Such a system is called a fuzzy associative memory (FAM) [3].

Fuzzy associative memories(FAMs) belong to the class of *fuzzy neural networks*(FNNs). An FNN is an artifical neural network (ANN) whose input patterns, output patterns and / or connection weights are fuzzy-valued [3]. Original FAMs was initiated in the early 1990s by approach of Kosko's FAM [7] also called *fuzzy morphological associative memories*(FMAMs). There are some simple examples of FMAMs such as Kosko's max-min and max-product FMAM, max-min FMAM of Junbo at al., [7, 6]. After FMAMs grew

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out of a gray-scale associative memory model called *morphological associative memory* (MAM) [9, 10]. In fact morphological associative memories perform elementary operations of mathematical morphology (MM) in a complete lattices. Now recently, MM was extended from complete lattices to complete inf-semilattices (cisl) by using reference function. A complete inf-semilattice is a set in which every arbitrary subset has an infimum (but not necessarily a supremum exist).

In this paper in section 3, we introduced a semilattice projection fuzzy associative memory (SL-PFAM) based on complete inf-semilattice derived from complete lattice $[0,1]^n$, which project to input pattern into the set of all max-C combination of the fundamental memory set. Finally, in experimental section we compare the performance of the SL-PFAMs using the min-D FMAM, max-C FMAM and max-C PFAM (C and D stand for conjunction and disjunction) as reference functions for gray scale images reconstruction.

2 Some Mathematical Background

Let $\mathbb{L}$ and $\mathbb{M}$ be complete lattices. Consider two arbitrary operators $\varepsilon : \mathbb{L} \to \mathbb{M}$ and $\delta : \mathbb{M} \to \mathbb{L}$ are erosion and dilation if and only if they commutes respectively with the infimum and supremum operations for every subset $X \subseteq \mathbb{L}$ and $Y \subseteq \mathbb{M}$:

$$\varepsilon\left(\bigwedge X\right) = \bigwedge_{x\in X} \varepsilon(x) \quad \text{and} \quad \delta\left(\bigvee Y\right) = \bigvee_{y\in Y} \delta(y). \tag{1}$$

Erosion and dilation are increasing operators, i.e., $\forall\, x, y \in \mathbb{L}$, if $x \leq y$ then $\varepsilon(x) \leq \varepsilon(y)$ and $\delta(x) \leq \delta(y), \forall x, y \in \mathbb{M}$. The erosion and dilation are related by the notion of adjunction [4, 11]. Thus the pair (δ, ε) is an adjunction from $\mathbb{L}$ to $\mathbb{M}$ or that ε and δ are adjoint if and only if we have

$$\delta(x) \leq y \Leftrightarrow x \leq \varepsilon(y)\ \forall\, x \in \mathbb{M}\ y \in \mathbb{L}. \tag{2}$$

If $\mathbb{L} = \mathbb{M}$, then (ε, δ) is called an adjunction on $\mathbb{L}$. The process of adjunction is of fundamental importance in mathematical morphology since it allows to define a unique dilation δ assoicated to a given erosion ε. Recently, Heijmans and Keshet [5] was developed *self-dual morphology* based on inf-semilattices. According to Heijmans and Keshet, the set $\mathbb{L}$ becomes a *complete inf-semilattice* (cisl) if we choose an arbitrary reference element $r \in \mathbb{L}$ and use the following partial order:

$$a \preceq_r b \Leftrightarrow r \vee a \leq b \vee r \textit{ and } b \wedge r \leq a \wedge r \tag{3}$$

The corresponding infimum in $(\mathbb{L}, \preceq_r)$ is

$$\curlywedge_{i\in I}{}_{\mathrm{r}} a_i = \left(r \wedge \bigvee_{i\in I} a_i\right) \vee \bigwedge_{i\in I} a_i = \left(r \vee \bigwedge_{i\in I} a_i\right) \wedge \bigvee_{i\in I} a_i \tag{4}$$

The ordering $\preceq_r$ coincides with the usual ordering in Boolean lattices. In case of complete inf-semilattices $(\mathbb{L}, \preceq_r)$, where the infimum $\bigwedge$ is defined but the supremum $\bigvee$ is not necessarily so, it is only possible to have the same side of reference function. The class of fuzzy sets in X will be denoted by $\mathcal{F}(X) = [0,1]^X$. Note that fuzzy set theory can be used for the design of image operator since an image $f : X \to [0,1]$ can be interpreted as a fuzzy set of X. From now on, an image $f \in \mathcal{F}(X)$ will be called fuzzy image. Since by the partial ordering on $[0,1]$ induces a partial ordering on $\mathcal{F}(X)$ also complete lattice. If we select a reference function $r \in \mathcal{F}(X)$ and use the partial ordering $f \preceq_r g$ define as $f(x) \preceq_r g(x)$ for all $x \in X$, then $(\mathcal{F}(X), \preceq_r)$ becomes a cisl.

The positive part $(f-r)^+$ and the negative part $(f-r)^-$ of $f-r$ for all $f, r \in \mathcal{F}(X)$ are respectively defined by $(f-r)^+ = (f-r) \vee 0$ and $(f-r)^- = -(f-r) \vee 0$, where 0 denotes the null element of the $\mathcal{F}(X)$. The elements $(f-r)^+$ and $(f-r)^-$ of the $\mathcal{F}(X)$ are disjoint fuzzy sets, i.e., $(f-r)^+ \wedge (f-r)^- = 0$. Let us recall the following partial order $\preceq_r$ on $\mathcal{F}(X)$ into a cisl [5].

Proposition 2.1. *Consider the binary relation $\preceq_r$ on $\mathcal{F}(X) = ([0,1]^{\mathbf{X}}, \preceq_r)$ that is defined as follows:*

$$f \preceq_r g \Leftrightarrow (f-r)^+ \preceq (g-r)^+ \, and \, (f-r)^- \preceq (g-r)^- \tag{5}$$

We have that $(\mathcal{F}(X), \preceq_r)$ is a cisl whose least element is r. The infimum of an arbitrary subset $\{f_i : i \in I\}$ of $\mathcal{F}(X)$ is given by

$$\curlywedge_{j\in J} {}_r f_j = \bigwedge_{j\in J} (f_j - r)^+ - \bigwedge_{j\in J} (f_j - r)^- + r. \tag{6}$$

Moreover precisely, (f_r) denotes $f - r$ for all $f, r \in \mathcal{F}(X)$. Note that $f \preceq_r g$ is equivalent to having both $(f_r)^+ \leq (g_r)^+$ and $(f_r)^- \leq (g_r)^-$. This observation leads to the above expression can be written as:

$$\curlywedge_{j\in J} {}_{\mathrm{r}} f_j = \bigwedge_{j\in J} (f_j)_r^+ - \bigwedge_{j\in J} (f_j)_r^- + r. \tag{7}$$

2.1 Some Basic operations of fuzzy set theory

Definition 2.1. *A* fuzzy conjunction *and* fuzzy disjunction *are define as increasing mappings $C, D : [0,1] \times [0,1] \to [0,1]$ that satisfies $C(0,0) = C(0,1) = C(1,0) = 0$ and $C(1,1) = 1$ and $D(0,0) = 0$ and $D(0,1) = D(1,0) = D(1,1) = 1$ respectively. The following operators are simple examples of fuzzy conjunction and fuzzy disjunction:*

$$C_M(x,y) = x \wedge y, \quad C_P(x,y) = x \cdot y, \quad C_L(x,y) = 0 \vee (x+y-1). \tag{8}$$
$$D_M(x,y) = x \vee y, \quad D_P(x,y) = x + y - x \cdot y, \quad D_L(x,y) = 1 \wedge (x+y). \tag{9}$$

Definition 2.2. *The binary operations $I, \bar{I} : [0,1] \times [0,1] \to [0,1]$ are called* fuzzy implication *and* fuzzy coimplication *if they are decreasing in the first argument, increasing in the second argument and satisfies the following conditions respectively:*

$$I(0,0) = I(1,1) = I(0,1) = 1 \quad and \quad I(1,0) = 0.$$
$$\bar{I}(0,0) = \bar{I}(1,1) = \bar{I}(1,0) = 0 \quad and \quad \bar{I}(0,1) = 1.$$

Some particular fuzzy implication and fuzzy coimplication, which were introduced by Gödel, Goguen, and Lukasiewicz can be found below [1] respectively:

$$I_M(x,y) = \begin{cases} 1 & if x \leqslant y \\ y & if x > y. \end{cases}, \; I_P(x,y) = 1 \wedge \left(\frac{y}{x}\right), \; I_L(x,y) = 1 \wedge (y - x + 1). \tag{10}$$

$$\bar{I}_M(x,y) = \begin{cases} 0 & x \geq y \\ y & x < y. \end{cases}, \; \bar{I}_P(x,y) = 0 \vee \left(\frac{y-x}{1-x}\right), \quad \bar{I}_L(x,y) = 0 \vee (y - x). \tag{11}$$

Furthermore, a fuzzy conjunction and a fuzzy implication, as well as fuzzy disjunction and a fuzzy coimplication can be related by means of a fundamental concept of mathematical morphology called *adjunction* [1].

Definition 2.3. *A fuzzy conjunction C and a fuzzy implication I form an* adjunction *if and only if $C(z,\cdot)$ and $I(\cdot,z)$ are* adjoint *for every $z \in [0,1]$ i.e.*

$$C(z,y) \leq x \;\Leftrightarrow\; y \leq I(x,z). \tag{12}$$

Similarly a pair $(D,\bar{I})$ forms an adjunction *if and only if $D(z,\cdot)$ and $\bar{I}(\cdot,z)$ are* adjoint *for every $z \in [0,1]$ i.e.,*

$$D(z,y) \geq x \;\Leftrightarrow\; y \geq \bar{I}(x,z). \tag{13}$$

Thus the pairs $(D_M,\bar{I}_M), (D_P,\bar{I}_P)$ and $(D_L,\bar{I}_L)$ are the examples of adjoint operators.

Let us recall the fuzzy matrix products that can be used to describe several FMAM models [13, 12]. The above fuzzy operations C, D,I and $\bar{I}$ can be combined with maximum or the minimum operation to yield the following matrix product. We recall the *max-C product* and *min-D product* of $A \in [0,1]^{m\times k}$ and $B \in [0,1]^{k\times n}$ are respectively denoted by $E = A \circ B$ and $G = A \bullet B$ as follows:

$$e_{ij} = \bigvee_{\xi=1}^{k} C(a_{i\xi}, b_{\xi j}), \text{ and } g_{ij} = \bigwedge_{\xi=1}^{k} D(a_{i\xi}, b_{\xi j}) \,\forall i = 1\ldots m, \; \forall j = 1\ldots n. \tag{14}$$

Similarly, the *min-I product* and the *max-$\bar{I}$ product* denoted by $H = A \circledast B$, $K = A \bar{\circledast} B$ respectively given by the following equations:

$$h_{ij} = \bigwedge_{\xi=1}^{k} I(a_{i\xi}, b_{\xi j}) \text{ and } k_{ij} = \bigvee_{\xi=1}^{k} \bar{I}(a_{i\xi}, b_{\xi j}) \,\forall\, i = 1,\ldots,m, \; \forall\, j = 1,\ldots,n. \tag{15}$$

Note that, $\bar{I}$ represent fuzzy coimplication that form adjunction with fuzzy disjunction D.

2.2 Brief review on the min-D FMAMs and the max-C FMAMs

2.2.1 Min-D FMAMs;

Let us recall the *min-D fuzzy morphological associative memories* corresponds to a single layer *fuzzy neural network* (FNN) with neurons that compute the minimum of fuzzy disjunction operation.

These neurons can be formulated in terms of *min-D fuzzy matrix products*. In mathematical form, the min-D FMAMs is a mapping $\mathcal{M} : [0,1]^n \rightarrow [0,1]^m$ determined by the following equation:

$$\mathbf{y} = \mathcal{M}(\mathbf{x}) = M \bullet \mathbf{x} \;\; \forall\, \mathbf{x} \in [0,1]^n. \tag{16}$$

where the symbol, " $\bullet$ " denote a min-D product and M represent the synaptic weight matrix, which is belong to $[0,1]^{m\times n}$. The resulting model will be called min-D FMAMs. Furthermore observe that, the operator $\mathcal{M}$ represent an erosion if and only if $D(\cdot, x)$ is an erosion for every $x \in [0,1]$ [13].

2.2.2 Max-C FMAMs, *the negation of min-D FMAMs*;

In this subsection we can also formulate, a max-C FMAMs by using max-C fuzzy matrix product. Let $W \in [0,1]^{m\times n}$denote the synaptic weight matrix. Given an arbitrary fuzzy input pattern $\mathbf{x} \in [0,1]^n$, then we compute the corresponding fuzzy output pattern $\mathbf{y} \in [0,1]^m$ as follows;

$$\mathbf{y} = \mathcal{W}(\mathbf{x}) = W \circ \mathbf{x} \;\; \forall\, \mathbf{x} \in [0,1]^n. \tag{17}$$

where the symbol, " $\circ$ " denote a max-C product as defined in Eq. (14). Thus the operator $\mathcal{W}$ yields a FMAM model if and only if the corresponding fuzzy conjunction corresponds to a dilation in the second argument. In this case, the associative mapping $\mathcal{W} : [0,1]^n \rightarrow [0,1]^m$ represents a dilation.

3 Semilattice based Projection Fuzzy Associative Memory (SL-PFAM)

In this section, we introduce a *semilattice projection fuzzy associative memory* (SL-PFAM) based on the cisl derived from $[0,1]^n$. For this let us recall the projection max-C fuzzy associative memory (PFAM) defined by Santos and Valle [2], i.e., the max-C PFAM which project to input pattern in the set

$$\mathcal{C}(X) = \left\{ \mathbf{z} = \bigvee_{\xi=1}^{k} C(\eta_\xi, \mathbf{x}^\xi) : \eta_\xi \in [0,1] \right\} \tag{18}$$

of all combination of max-C of the fundamental memories set $X = \{\mathbf{x}^1, \ldots, \mathbf{x}^k\} \subset [0,1]^n$. Thus a max-$C$ PFAM is a mapping $\mathcal{V} : [0,1]^n \rightarrow [0,1]^n$ determined by the following equation:

$$\mathcal{V}(\mathbf{x}) = \bigvee \{\mathbf{z} \in \mathcal{C}(X) : \mathbf{z} \leq \mathbf{x}\} \quad \forall\, \mathbf{x} \in [0,1]^n. \tag{19}$$

Such as the min-D FMAM, the PFAM also exhibit perfect recall of corrupted or incomplete version of patterns if and only if the fuzzy conjunction is associative as well as has a right identity. The advantages of $\mathcal{V}$ has excellent absolute storage capacity, one step convergence if employed with feedback. On the other hand, the PFAM is more suited for the reconstruction of patterns corrupted by dilative noise, but it is unable to deal with arbitrary noise. Consider adjoint operators C and I, then can be expressed the outputs $\mathcal{V}(\mathbf{x})$ and η_ξ by means of max-C and min-I products as following;

$$\mathcal{V}(\mathbf{x}) = \bigvee_{\xi=1}^{k} C\left(\eta_\xi, \mathbf{x}^\xi\right) \quad \text{such that} \quad \eta_\xi = \bigwedge_{j=1}^{n} I(\mathbf{x}_j^\xi, \mathbf{x}_j)\, \forall \xi = 1, \ldots, k. \tag{20}$$

for input pattern $\mathbf{x} \in [0,1]^n$. More generally, we define the following theorem which represent a SL-PFAM based on cisl derived from $[0,1]^n$ by using reference function $\rho : [0,1]^n \longrightarrow [0,1]^n$, that is guaranteed to yield perfect recall of the original patterns if the reference function recall the original patterns.

Theorem 3.1. *Consider an adjunction (C, I) and fuzzy conjunction is associative as well as has a right identity. Given a fundamental memory set $X = \{\mathbf{x}^1, \ldots, \mathbf{x}^k\} \subset [0,1]^n$. Thus a SL-PFAM $\mathcal{V}_\rho(\mathbf{x}) : [0,1]^n \to [0,1]^n$ based on semilattices be define as:*

$$\mathcal{V}_\rho(\mathbf{x}) = \mathcal{V}(\mathbf{x} - \rho(\mathbf{x}))^+ - \mathcal{V}(\mathbf{x} - \rho(\mathbf{x}))^- + \rho(\mathbf{x}) \quad \forall\, \mathbf{x} \in [0,1]^n. \tag{21}$$

If $\rho(\mathbf{x}) = \mathbf{x}^\xi$ for all $\xi = 1, \ldots, k$. Thus $\mathcal{V}_\rho$ yields perfect recall of the original patterns i.e., $\mathcal{V}_\rho(\mathbf{x}^\xi) = \mathbf{x}^\xi\ \forall \xi = 1, \ldots, k$. For an arbitrary input pattens $\mathbf{x} \in [0,1]^n$, the output pattern satisfying the following relationship;

$$\rho(\mathbf{x}) \preceq_{\rho(\mathbf{x})} \mathcal{V}_\rho(\mathbf{x}) \preceq_{\rho(\mathbf{x})} \mathbf{x}\,. \tag{22}$$

Note that, in particular the $\mathcal{M}$, $\mathcal{W}$ and $\mathcal{V}$ used as reference functions in following subsection.

3.1 Simulations in Gray-Scale Images Reconstruction

In this section we perform some experiments using the eight images i.e., (Lena, Cameraman, Airplane, House, Vehicle, Boat, Church and Watch) that are available at the internet site of the Mathematical Imaging and Computational Intelligence Laboratory, University of Campinas [8]. These images have size 128×128 and 256 gray levels. In this simulations, we converted these images into ten fuzzy images by normalizing the respective pixel values within the range $[0, 1]$. For each of these images, we generated a vector $\mathbf{u}^\xi, \in [0,1]^{16384}$ for $\xi = 1, \ldots, 8$.

In this experiment, we probed the SL-PFAM, i.e., $\mathcal{V}_\rho$ with reference functions (choice) as the min-D FMAM $\mathcal{M}$, the max-C FMAM $\mathcal{W}$ and $\mathcal{V}$, with the noisy, incomplete and distorted patterns represented by $\tilde{\mathbf{u}}^1, \ldots, \tilde{\mathbf{u}}^7$ as shown in the top row of Fig.1. The first two images cameraman and lena in the top row of Fig.1 corrupted by pepper noise and salt noise with probability 25% respectively. The next three images in that row represent incomplete versions of the original images. The last two images corrupted by salt and pepper noise with probability 20% and additive Gaussian noise with mean 0 and variance 0.05. The below row of Fig.1 depicts the outputs produced by PFAM $\mathcal{V}^L_\rho$ with aforementioned reference function. The Table 1 show the numerical results of $\mathcal{V}^L_\mathcal{V}$, $\mathcal{V}^L_\mathcal{M}$ and $\mathcal{V}^L_\mathcal{W}$ such as slightly better then $\mathcal{V}$, $\mathcal{M}$ and $\mathcal{W}$ (reference functions), but the numerical results of these reference functions not shown due to the space limitation.

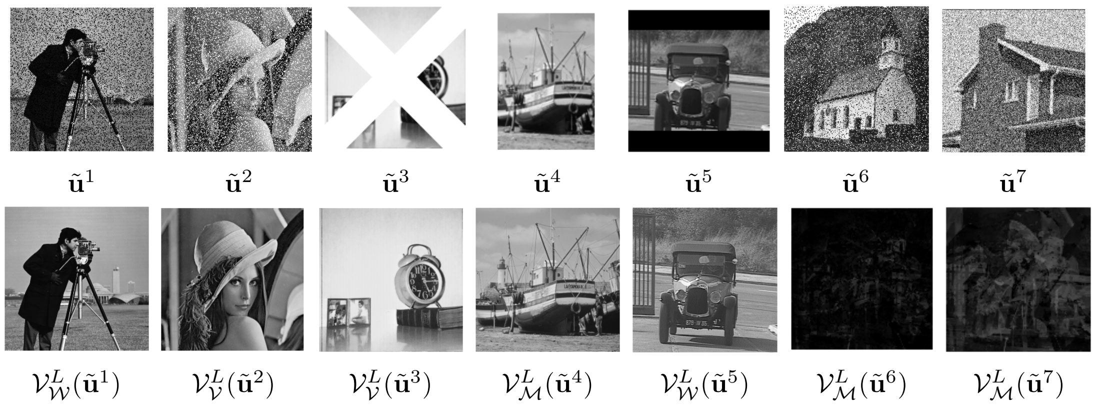

Figure 1: The top row shows distorted, incomplete and corrupted versions of the original images. The bottom rows depict the images that were recalled by $\mathcal{V}^L_\mathcal{V}$, $\mathcal{V}^L_\mathcal{W}$ and $\mathcal{V}^L_\mathcal{M}$.

Table 1: NMSE produced by SL-PFAM Models in applications to the original patterns.

Input pattern:	$\tilde{\mathbf{u}}^1$	$\tilde{\mathbf{u}}^2$	$\tilde{\mathbf{u}}^3$	$\tilde{\mathbf{u}}^4$	$\tilde{\mathbf{u}}^5$	$\tilde{\mathbf{u}}^6$	$\tilde{\mathbf{u}}^7$
SL-PFAM-$\mathcal{V}^L_{\mathcal{V}}$	1.0000	**0.0002**	**0.1162**	0.1134	0.9670	0.9699	0.9227
SL-PFAM-$\mathcal{V}^L_{\mathcal{W}}$	**0.0097**	1.0000	1.0820	1.0047	**0.1291**	1.0473	0.8716
SL-PFAM-$\mathcal{V}^L_{\mathcal{M}}$	0.8532	0.0098	0.1222	**0.1115**	0.8193	**0.9178**	**0.8021**

4 Conclusions

We presented a projection fuzzy associative memory on semilattice derived from complete lattice $[0,1]^n$ using reference function, with an application for the reconstruction of gray-scale images.

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