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A note on sub-orthogonal lattices

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Abstract. It is shown that, given any k-dimensional lattice Λ, there is a lattice sequence Λ_w, $w \in \mathbb{Z}$, with sub-orthogonal lattice $\Lambda_o \subset \Lambda$, converging to Λ (unless equivalence), also we discuss the conditions for faster convergence.

Keywords. Sub-orthogonal Lattice, Dense Packing, Spherical Code.

1 Introduction

A large class of problems in coding theory are related to the study lattices having a sub-lattices with orthogonal basis (sub-orthogonal lattices). Several authors investigated the relationship of sub-orthogonal with spherical codes, and with q-ary codes (see [1, 3, 8–10, 13, 15–18]), but of course that does not restrict to these problems ([2, 4, 5, 12]).

In general, lattice problems concentrated in determining certain parameters, such as the shortest vector, packing radius and packing density; radius cover and cover density. The points of are interpreted as elements of a code, thereby determining a coding scheme and efficient decoding is essential. There are several buildings in the literature that establish the relationship of linear codes with lattices ([7]).

This paper is organized as follows. We will fix the notations and definitions of lattices in Section 2. We present the construction of a sequence of lattices in section 3. In Section 4, we present a case study for special cases lattices for: the root lattices $\mathbb{D}_n$ and $\mathbb{E}_n$ $(n = 7, 8)$ and Leech lattice Λ_{24}.

1.1 Background definitions and results

A *lattice* in $\mathbb{R}^n$ is an discrete additive subgroup of $\mathbb{R}^n$, Λ, which has a *generator matrix* with full rank, $n \times k$, $\mathcal{B}$, e.g, $\boldsymbol{v} \in \Lambda \leftrightarrow \boldsymbol{v} = \boldsymbol{u}^t\mathcal{B}$ ($\boldsymbol{u} \in \mathbb{Z}^k$, k is said rank of Λ. The *determinant* of a lattice is $\det(\Lambda) = \det(\mathcal{G})$, there $\mathcal{G} = \mathcal{B}\mathcal{B}^t$ is an *Gram matrix* of lattice Λ and the *volume* of lattice is $\sqrt{\det(\Lambda)}$ (volume of the parallelotope generate for rows of $\mathcal{B}$). The *minimum norm* of Lattice Λ, $\rho(\Lambda)$, is $\min\{\|\boldsymbol{v}\|; \boldsymbol{v} \in \Lambda \text{ and } \boldsymbol{v} \neq \boldsymbol{0}\}$ and *center density packing* of Λ is $\delta_\Lambda = \dfrac{\rho(\Lambda)^n}{2^n \operatorname{vol}(\Lambda)}$. Two lattices Λ_1 and Λ_2, with generator matrices $\mathcal{B}_1$ and $\mathcal{B}_2$ are *equivalence* if, only if $\mathcal{B}_1 = c\,\mathcal{U}\mathcal{B}_2\mathcal{O}$, there $c \in \mathbb{R}$, $\mathcal{U}$ is *unimodular matrix* (integer, $k \times k$ matrix with $\det(\mathcal{U}) = \pm 1$) and $\mathcal{O}$ is the orthogonal, $n \times n$ matrix

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($\mathcal{O}\mathcal{O}^t = \mathcal{I}_n$, $\mathcal{I}_n$ identity matrix $n \times n$). *Dual lattice* of Λ is a lattice, Λ^*, obtained for all vectors $\boldsymbol{u} \in \text{span}(\mathcal{B})$ (there $\text{span}(\mathcal{B})$ is a vector space generated by the rows of $\mathcal{B}$) with that $\boldsymbol{u} \cdot \boldsymbol{v} \in \mathbb{Z}, \forall \boldsymbol{v} \in \Lambda$, the generator matrix of Λ^* is $\mathcal{B}^* = (\mathcal{B}\mathcal{B}^t)^{-1}\mathcal{B}$, in particular, $\mathcal{B}^* = \mathcal{B}^{-t}$ if $n = k$. A sub-lattice, Λ', is a subset of Λ which is also lattice, if Λ' has generator matrix is formed by orthogonal row vectors we will say that it is a sub-orthogonal.

Since the lattice is a group, remember that the quotiente of the lattice Λ by sublattice Λ', $\frac{\Lambda}{\Lambda'}$, is as a finite abelian group with M elements, where M is a ratio of volume of sublattice Λ' by the volume of lattice Λ, e.g., $M = \frac{\text{vol}(\Lambda')}{\text{vol}(\Lambda)}$. The M elements of lattice Λ, can be seen as an orbit of null vector in k-dimensional torus $\frac{\Lambda}{\Lambda'}$. This essentially establishes the relationship with a central spherical class codes, as well as a class of linear codes track construction "A" and similar constructions, see more details ([7]).

2 Suborthogonal sequences

Consider a lattice $\Lambda \subset \mathbb{R}^n$, of rank n, contain a the orthogonal sub-lattice, $\Lambda_o \subset \Lambda$, such that Λ_o is equivalence to $\mathbb{Z}^n$, e.g., the generator matrix of Λ_o is $c\,\mathcal{O}$, with $\mathcal{O}\mathcal{O}^t = \mathcal{I}_n$. Let $\mathcal{B}$ and $\mathcal{B}^* = \mathcal{B}^{-t}$ the generator matrices of Λ and Λ^* (respectively). Assuming that $\mathcal{B}^*$ has integer entries, Then lattice, Λ, with generator matrix $\mathcal{B} = \text{adj}(\mathcal{B}^*) = \det(\mathcal{B}^*)\mathcal{B}^{*-t}$ has a sub-orthogonal lattice, Λ_o, with generator matrix $\mathcal{B}^*\mathcal{B} = \det(\mathcal{B}^*)\mathcal{I}_n$. The ratio of volume measured quantities points and in this case it is $\frac{\text{vol}(\Lambda_o)}{\text{vol}(\Lambda)} = \frac{\det(\mathcal{B}^*\mathcal{B})}{\det(\mathcal{B})} = \det(\mathcal{B}^*)$.

In general, we want to build code with many points and as we increase the amount of points the lattice come on, unless of equivalence, a similar lattice to a previously chosen. This motivates the following construction:

Proposição 2.1. *Let's Λ be a lattice whose dual of the generator matrix $\mathcal{B}^*$ has integer entries and Λ^*_w with matrix generator $\mathcal{B}^*_w = w\mathcal{B}^* + \mathcal{P}$, where $\mathcal{B}^*$, is generator matrix of equivalent lattice of Λ^*, $\mathcal{P}$ is an integer matrix any and w is integer. Then lattices Λ^*_w and Λ_w with generator matrices $\mathcal{B}^*_w$ and $\mathcal{B}_w = \text{adj}(\mathcal{B}^*_w)$ (respectively) to satisfy $\frac{1}{w}\Lambda^*_w \longrightarrow^{w\to\infty} \Lambda^*$ and by continuity of the matrix inversion process $\frac{1}{\det(\frac{1}{w}\mathcal{B}^*_w)}\Lambda_w \longrightarrow^{w\to\infty} \Lambda$.*

Corolário 2.1. *Let's Λ be a lattice whose dual of the generator matrix $\mathcal{B}^*$ and Λ^*_w with matrix generator $\mathcal{B}^*_w = w\mathcal{B}^* + \mathcal{P}$, where $\mathcal{B}^*$, is generator matrix of equivalent lattice of Λ^*, $\mathcal{P} = \lfloor w\mathcal{B}^* \rceil - w\mathcal{B}^*$, in other words $\mathcal{B}^*_w = \lfloor w\mathcal{B}^* \rceil$ (rounded entries) . Then lattices Λ^*_w and Λ_w with generator matrices $\mathcal{B}_w$ and $\mathcal{B}_w = \text{adj}(\mathcal{B}^*_w)$ (respectively) to satisfy $\frac{1}{w}\Lambda^*_w \longrightarrow^{w\to\infty} \Lambda^*$ and by continuity of the matrix inversion process $\frac{1}{\det(\frac{1}{w}\mathcal{B}^*_w)}\Lambda_w \longrightarrow^{w\to\infty} \Lambda$.*

The corollary allows to extend the use of the proposition for whose lattices dual have not, un less equivalence, integer generator matrix.

When it comes to convergence, it's natural curiosity with regard to convergence, this motivates the following propositions:

Proposição 2.2 (faster dual convergence)**.** *Let's Λ^*_w and Λ_w as in Proposition 2.1 Then faster convergence is obtained by imposing $\mathcal{P} = \mathcal{S}\ \mathcal{B}$, where $\mathcal{S}$ is antisymmetric matrix $n \times n$ and minimizing inputs $\mathcal{P}\mathcal{P}^t$, naturally $\mathcal{P}$ identically zero is the best convergence, because there is no error.*

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Proposição 2.3 (faster convergence). *Let's Λ_w^* and Λ_w as in Proposition 2.1 Then faster convergence is obtained by imposing $\mathcal{P} = \mathcal{B}^*\mathcal{S}$, where $\mathcal{S}$ is antisymmetric matrix $n \times n$ and minimizing inputs $\mathcal{B}\mathcal{P}^t\mathcal{G}\mathcal{P}\mathcal{B}^t$, naturally $\mathcal{P}$ identically zero is the best convergence, because there is no error.*

The structure of the group obtained by the quotient of lattice sequence, Λ_w, by their respective orthogonal sub-lattice can be determined and extended, applying the Theorem 2.4.13 [H. Cohen book pp 75].

In particular, $\mathcal{B}^*$ is lower triangular matrix and $\mathcal{P} = \mathcal{C}_n = (c_{i,j})$ (cyclic perturbation), where $c_{i,j} = 1$ if $j = i + 1$ and $c_{i,j} = 0$ otherwise, the quotient is cyclic group although convergence is not nearly quadratic.

Lattices of rank n that unless equivalence are sub-lattices the integer lattice $\mathbb{Z}^n$, play an interesting role with regard to convergence as discussed below with case study, next section.

3 Case study

In this section we present the construction applied to special cases: $\mathbb{D}_n$, $\mathbb{E}_n (n = 7, 8)$, and Λ_{24}. Illustrate the performance of the quadratic ($\mathcal{P} = \mathbf{0}_n$ null matrix $n \times n$) versus quadratic convergence associated with groups of no more than two generators ($\mathcal{P} = \mathcal{P}_n$, good pertubation $n \times n$), it is unfortunately not possible to obtain a dimension anyone quotient that is cyclic and at the same time has quadratic convergence. We will display also results showing the performance of this construction with limiting associated spherical codes proposed in the paper [13], the problem was partially resolved in the previous article ([1]), but the solution is presented only for special lattices and the solution in each case depends on many calculations explored by sub-orthogonal lattice and this article will not explore the concept of initial vector.

3.1 The root lattice $\mathbb{D}_n$ $(n \geq 3)$

We consider the generate matrix of $\mathbb{D}_n^*$ as $\mathcal{D}_n^*$ and good perturbation is $\mathcal{P}_n$:

$$\mathcal{D}_n^* = \begin{bmatrix} 2 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix} \text{ and } \mathcal{P}_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 1 \\ -1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 1 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}.$$

The good perturbation is $\mathcal{P}_n$ as above, this case the quotient is cyclic case odd n, the performance is illustrated in Table 1 and in Table 2

3.2 The root lattice $\mathbb{E}_n$

Unless equivalence assuming that $\mathbb{E}_7^*$, $\mathbb{E}_{8,1}^*$ and $\mathbb{E}_{8,2}^*$ are generated by matrices $\mathcal{E}_7^*$, $\mathcal{E}_{8,1}^*$ and $\mathcal{E}_{8,2}^*$ and the good perturbation $\mathcal{P}_7$, $\mathcal{P}_{8,1}$ and $\mathcal{P}_{8,2}$.

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Table 1: 3-dimensional performance, for perturbations $\mathbf{0}_n$, $\mathcal{P}_n$ and $\mathcal{C}_n$ respectively.

$M(w)$	$\delta(\Lambda_w)$	Group	$M(w)$	$\delta(\Lambda_w)$	Group	$M(w)$	$\delta(\Lambda_w)$	Group
4	0.176777	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	7	0.133631	$\mathbb{Z}_7$	3	0.0721688	$\mathbb{Z}_3$
32	0.176777	$\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4$	38	0.162221	$\mathbb{Z}_{38}$	26	0.0969021	$\mathbb{Z}_{26}$
108	0.176777	$\mathbb{Z}_3 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_6$	117	0.169842	$\mathbb{Z}_{117}$	93	0.116923	$\mathbb{Z}_{93}$
256	0.176777	$\mathbb{Z}_4 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_8$	268	0.172774	$\mathbb{Z}_{268}$	228	0.129349	$\mathbb{Z}_{228}$
500	0.176777	$\mathbb{Z}_5 \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{10}$	515	0.174183	$\mathbb{Z}_{515}$	455	0.137602	$\mathbb{Z}_{455}$
864	0.176777	$\mathbb{Z}_6 \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{12}$	882	0.174964	$\mathbb{Z}_{882}$	798	0.143442	$\mathbb{Z}_{798}$
1372	0.176777	$\mathbb{Z}_7 \oplus \mathbb{Z}_{14} \oplus \mathbb{Z}_{14}$	1393	0.175439	$\mathbb{Z}_{1393}$	1281	0.147780	$\mathbb{Z}_{1281}$
2048	0.176777	$\mathbb{Z}_8 \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_{16}$	2072	0.175750	$\mathbb{Z}_{2072}$	1928	0.151126	$\mathbb{Z}_{1928}$
2916	0.176777	$\mathbb{Z}_9 \oplus \mathbb{Z}_{18} \oplus \mathbb{Z}_{18}$	2943	0.175964	$\mathbb{Z}_{2943}$	2763	0.153783	$\mathbb{Z}_{2763}$
4000	0.176777	$\mathbb{Z}_{10} \oplus \mathbb{Z}_{20} \oplus \mathbb{Z}_{20}$	4030	0.176117	$\mathbb{Z}_{4030}$	3810	0.155943	$\mathbb{Z}_{3810}$

Table 2: 3 to 6-dimensional performance, for perturbations $\mathcal{P}_n$.

$\frac{\delta(\Lambda_w)}{\delta(\mathbb{D}_3)}$	Group	$\frac{\delta(\Lambda_w)}{\delta(\mathbb{D}_4)}$	Group	$\frac{\delta(\Lambda_w)}{\delta(\mathbb{D}_5)}$	Group	$\frac{\delta(\Lambda_w)}{\delta(\mathbb{D}_6)}$	Group
0.7559	$\mathbb{Z}_7$	1.	$\mathbb{Z}_3 \oplus \mathbb{Z}_6$	0.6718	$\mathbb{Z}_{41}$	0.675	$\mathbb{Z}_{10} \oplus \mathbb{Z}_{10}$
0.9177	$\mathbb{Z}_{38}$	1.	$\mathbb{Z}_9 \oplus \mathbb{Z}_{18}$	0.8732	$\mathbb{Z}_{682}$	0.8576	$\mathbb{Z}_{17} \oplus \mathbb{Z}_{170}$
0.9608	$\mathbb{Z}_{117}$	1.	$\mathbb{Z}_{19} \oplus \mathbb{Z}_{38}$	0.9371	$\mathbb{Z}_{4443}$	0.9269	$\mathbb{Z}_{74} \oplus \mathbb{Z}_{370}$
0.9774	$\mathbb{Z}_{268}$	1.	$\mathbb{Z}_{33} \oplus \mathbb{Z}_{66}$	0.9631	$\mathbb{Z}_{17684}$	0.9565	$\mathbb{Z}_{65} \oplus \mathbb{Z}_{2210}$
0.9853	$\mathbb{Z}_{515}$	1.	$\mathbb{Z}_{51} \oplus \mathbb{Z}_{102}$	0.9759	$\mathbb{Z}_{52525}$	0.9714	$\mathbb{Z}_{202} \oplus \mathbb{Z}_{2626}$
0.9897	$\mathbb{Z}_{882}$	1.	$\mathbb{Z}_{73} \oplus \mathbb{Z}_{146}$	0.9831	$\mathbb{Z}_{128766}$	0.9799	$\mathbb{Z}_{145} \oplus \mathbb{Z}_{10730}$
0.9924	$\mathbb{Z}_{1393}$	1.	$\mathbb{Z}_{99} \oplus \mathbb{Z}_{198}$	0.9875	$\mathbb{Z}_{275807}$	0.9851	$\mathbb{Z}_{394} \oplus \mathbb{Z}_{9850}$
0.9942	$\mathbb{Z}_{2072}$	1.	$\mathbb{Z}_{129} \oplus \mathbb{Z}_{258}$	0.9904	$\mathbb{Z}_{534568}$	0.9885	$\mathbb{Z}_{257} \oplus \mathbb{Z}_{33410}$
0.9954	$\mathbb{Z}_{2943}$	1.	$\mathbb{Z}_{163} \oplus \mathbb{Z}_{326}$	0.9924	$\mathbb{Z}_{959409}$	0.9909	$\mathbb{Z}_{650} \oplus \mathbb{Z}_{26650}$
0.9963	$\mathbb{Z}_{4030}$	1.	$\mathbb{Z}_{201} \oplus \mathbb{Z}_{402}$	0.9938	$\mathbb{Z}_{1620050}$	0.9926	$\mathbb{Z}_{401} \oplus \mathbb{Z}_{81002}$

$$\mathcal{E}_7^* = \begin{bmatrix} 1&0&0&0&-1&0&1\\ 0&1&0&0&-1&-1&1\\ 0&0&1&0&-1&-1&0\\ 0&0&0&1&0&-1&-1\\ 0&0&0&0&2&0&-2\\ 0&0&0&0&0&2&0\\ 0&0&0&0&0&0&2 \end{bmatrix}, \mathcal{E}_{8,1}^* = \begin{bmatrix} 1&0&0&0&-1&0&1&1\\ 0&1&0&0&-1&-1&1&0\\ 0&0&1&0&-1&-1&0&1\\ 0&0&0&1&0&-1&-1&-1\\ 0&0&0&0&2&0&-2&-2\\ 0&0&0&0&0&2&0&0\\ 0&0&0&0&0&0&2&0\\ 0&0&0&0&0&0&0&2 \end{bmatrix}, \mathcal{E}_{8,2}^* = \begin{bmatrix} 1&1&1&1&1&1&1&-7\\ -1&1&1&1&1&1&1&-5\\ 0&0&2&2&2&2&2&-10\\ 0&0&0&2&2&2&2&-8\\ 0&0&0&0&2&2&2&-6\\ 0&0&0&0&0&2&2&-4\\ 0&0&0&0&0&0&2&-2\\ 0&0&0&0&0&0&0&4 \end{bmatrix},$$

$$\mathcal{P}_7 = \begin{bmatrix} 0&1&0&0&0&0&0\\ -1&0&1&0&0&0&0\\ 0&-1&0&0&0&0&0\\ -1&-1&-1&0&-1&0&0\\ 0&0&-1&1&-1&0&-1\\ 0&0&1&1&1&0&1\\ 1&0&0&-1&1&-1&0 \end{bmatrix}, \mathcal{P}_{8,1} = \begin{bmatrix} 0&1&1&0&0&0&0&0\\ -1&0&0&0&0&0&0&0\\ 0&1&0&-1&0&0&-1&0\\ 0&0&0&0&0&0&0&0\\ 0&-1&-1&0&0&0&1&-1\\ 0&0&1&1&1&0&1&0\\ -1&-1&0&0&0&-1&0&1\\ 1&1&0&-1&0&0&-1&0 \end{bmatrix}, \mathcal{P}_{8,2} = \begin{bmatrix} -1&0&0&0&0&0&1&0\\ -1&-1&0&0&0&0&0&0\\ -1&-1&-1&0&1&0&0&0\\ -1&0&-1&-1&0&1&0&0\\ -1&0&0&-1&0&0&0&0\\ -1&-1&1&-1&0&-1&1&0\\ -1&-1&0&0&1&-1&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix}.$$

The performance is illustrated in Table 3 (note that the density ratio is deployed close and the amount of associated points are: 1.664.641.200 points for dual lattice $10\,\mathbb{E}_{8,1}^*+\mathcal{P}_{8,1}$ and 11.430.630.576 for dual lattice $9\,\mathbb{E}_{8,2}^* + \mathcal{P}_{8,2}$ (very more points in the second case). The Table 4 illustrates the performance applied in spherical codes, details in [13], the non-null perturbation is better in the case of $\mathbb{E}_{8,1}$ representation, moreover, point out that

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the performance is similar to the second representation with null perturbation (in bold: distances near and number of nearby points)

Table 3: 7 to 8-dimensional performance, for representations $\mathbb{E}_7^*$, $\mathbb{E}_{8,1}^*$ $\mathbb{E}_{8,2}^*$ and perturbations $\mathcal{P}_7$, $\mathcal{P}_{8,1}$ and $\mathcal{P}_{8,2}$.

$\frac{\delta(\Lambda_w)}{\delta(\mathbb{E}_7)}$	Group	$\frac{\delta(\Lambda_w)}{\delta(\mathbb{E}_{8,1})}$	Group	$\frac{\delta(\Lambda_w)}{\delta(\mathbb{E}_{8,2})}$	Group
0.2346	$\mathbb{Z}_2 \oplus \mathbb{Z}_{68}$	0.1204	$\mathbb{Z}_2 \oplus \mathbb{Z}_{78}$	0.2706	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{364}$
0.4161	$\mathbb{Z}_2 \oplus \mathbb{Z}_{1552}$	0.4022	$\mathbb{Z}_4 \oplus \mathbb{Z}_{2316}$	0.5065	$\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{15128}$
0.5966	$\mathbb{Z}_2 \oplus \mathbb{Z}_{15468}$	0.622	$\mathbb{Z}_6 \oplus \mathbb{Z}_{26154}$	0.6918	$\mathbb{Z}_2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{189252}$
0.7208	$\mathbb{Z}_2 \oplus \mathbb{Z}_{92192}$	0.7521	$\mathbb{Z}_8 \oplus \mathbb{Z}_{165912}$	0.7993	$\mathbb{Z}_2 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_{1251376}$
0.8005	$\mathbb{Z}_2 \oplus \mathbb{Z}_{391540}$	0.8284	$\mathbb{Z}_{10} \oplus \mathbb{Z}_{729030}$	0.8616	$\mathbb{Z}_2 \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{5612060}$
0.8522	$\mathbb{Z}_2 \oplus \mathbb{Z}_{1313328}$	0.8754	$\mathbb{Z}_{12} \oplus \mathbb{Z}_{2495268}$	0.8997	$\mathbb{Z}_2 \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{19429704}$
0.8869	$\mathbb{Z}_2 \oplus \mathbb{Z}_{3708572}$	0.9059	$\mathbb{Z}_{14} \oplus \mathbb{Z}_{7137186}$	0.9243	$\mathbb{Z}_2 \oplus \mathbb{Z}_{14} \oplus \mathbb{Z}_{55966708}$
0.911	$\mathbb{Z}_2 \oplus \mathbb{Z}_{9191488}$	0.9266	$\mathbb{Z}_{16} \oplus \mathbb{Z}_{17842224}$	0.9411	$\mathbb{Z}_2 \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_{140558432}$
0.9284	$\mathbb{Z}_2 \oplus \mathbb{Z}_{20572452}$	0.9412	$\mathbb{Z}_{18} \oplus \mathbb{Z}_{40176702}$	**0.9529**	$\mathbb{Z}_2 \oplus \mathbb{Z}_{18} \oplus \mathbb{Z}_{317517516}$
0.9412	$\mathbb{Z}_2 \oplus \mathbb{Z}_{42432080}$	**0.952**	$\mathbb{Z}_{20} \oplus \mathbb{Z}_{83232060}$	0.9615	$\mathbb{Z}_2 \oplus \mathbb{Z}_{20} \oplus \mathbb{Z}_{659296120}$

3.3 The Leech lattices Λ_{24}

The laminate lattice is generally dense in their respective dimensions in special dimensions, $n = 9, 15, 16, 19, 20, 21, 24$ admit integer representation un less equivalence, and in these cases can analyse the fast convergence, consider $n = 24$ the matrix generator of Leech Lattice, unless equivalence is $\mathfrak{L}_{24,1}$ and $\mathfrak{L}_{24,1}$, obtained as sub-lattice of $\mathbb{E}_1^8$ and $\mathbb{E}_2^8$ (respectively). Consider here $\mathfrak{L}_{24,1}^* = 4\mathfrak{L}_{24,1}^{-t}$ and $\mathfrak{L}_{24,2}^* = 8\mathfrak{L}_{24,2}^{-t}$.

We know that the Leech lattice can be regarded as a sub-lattice of the lattice $\mathbb{E}^8 \times \mathbb{E}^8 \times \mathbb{E}^8$, as in the 8-dimensional case: we use the non-null perturbation the first case ($\mathcal{P}_{24,1} = \begin{bmatrix} \mathcal{P}_{8,1} & 0 & 0 \\ 0 & \mathcal{P}_{8,1} & 0 \\ 0 & 0 & \mathcal{P}_{8,1} \end{bmatrix}$) and the second case to null perturbation ($\mathcal{P}_{24,2} = \boldsymbol{O}$) and we analyse the performance point of view of the spherical codes, see Table 5 (the first two columns refer to the first case).

4 Conclusions

We conclude that all n-dimensional lattice, up less scale, can be approximated by a sequence of lattices that have orthogonal sub-lattice. Furthermore, there is a degree of freedom $(n(n-1)/2)$ for quadratic convergence, this freedom induces quotient group with different number of generators and can make convergency more fast in certain applications, for example in the case of spherical codes are reticulated target has some multiple minimum vectors of some canonical vector, we find a non-null pertubation as it will be more efficient. We present here a method for finding lattices with sub-orthogonal, our method is simpler, more general and more efficient than the one presented in [1].

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Table 4: Show spherical code performance 8-dimensional case, for different representations.

	$\mathbb{E}_{8,1}$		$\mathbb{E}_{8,2}$	
	Distance	M − Points	Distance	M − Points
0	0.707107	4096	0.707107	65536
	0.707107	104976	0.500000	1679616
	0.500000	1048576	0.382683	16777216
	0.415627	6250000	**0.309017**	**100000000**
	0.366025	26873856	0.258819	429981696
	0.306802	92236816	0.222521	1475789056
	0.270598	268435456	0.195090	4294967296
	0.241845	688747536	0.173648	11019960576
	0.218508	1600000000	0.156434	25600000000
$\mathcal{P}_{8,i}$	0.839849	9264	0.639702	121024
	0.641669	156924	0.468092	2271024
	0.509472	1327296	0.366403	20022016
	0.419589	7290300	0.299852	112241200
	0.355527	29943216	0.253223	466312896
	0.307914	**99920604**	0.218878	1567067824
	0.271283	285475584	0.192596	4497869824
	0.242296	723180636	0.171869	11430630576
	0.218821	1664641200	0.155124	26371844800

References

[1] C. Alves, S.I.R. Costa, Commutative group codes in R^4, R^6, R^8 and R^{16} approaching the bound, *Discrete Math.*, 313(16):1677-1687, 2013.

[2] M. Berstein, N.J.A. Sloane, P.E. Wright, On sublattices of the hexagonal lattice, *Discrete Mathematics* 170:29-39, 1997.

[3] E. Biglieri, M. Elia, Cyclic-group codes for the Gaussian channel, *IEEE Transactions on Information Theory*, 22:624-629, 1976.

[4] A. Campello, J.E. Strapasson, On sequences of projections of the cubic lattice, *Comput. Appl. Math.*, 32(1):57-69, 2013.

[5] A. Campello, J.E. Strapasson, S.I.R. Costa, On projections of arbitrary lattices, *Linear Algebra Appl.*, 439(9):2577-2583, 2013.

[6] H.C. Cohen, *A Course in Computational Algebraic Number Theory*, Springer-Verlag, New York, 1993.

[7] J.H. Conway, N.J.A. Sloane, *Sphere Packings, Lattices and Groups*, Springer-Verlag, New York, 1999.

[8] S.I.R. Costa, M. Muniz, E. Augustini, R. Palazzo, Graphs, tessellations and perfect codes on flat tori, *IEEE Transactions on Information Theory*, 50:2363-2377, 2004.

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Table 5: Spherical code performance 24-dimensional case, for two different representations.

$\log_{10} M$	distance	$\log_{10} M$	distance
10.1917	0.633946	10.8371	0.57735
15.5128	0.484887	18.0618	0.408248
19.1994	0.370468	22.288	0.288675
21.9813	0.294144	25.2865	0.220942
24.2006	0.24225	27.6124	0.178411
26.0413	0.205264	29.5127	0.149429
27.6113	0.177774	31.1194	0.128473
28.9791	0.156625	32.5112	0.112635
30.1901	0.13989	33.7389	0.100256
31.2763	0.126336	34.8371	0.0903175
32.2609	0.115147	35.8305	0.0821655
33.161	0.10576	36.7374	0.0753593
33.99	0.097775	37.5717	0.0695919

[9] I. Ingemarsson, Group codes for the Gaussian channel, In *Lecture Notes in Control and Information Sciences*, 128:73-108, Springer Verlag, 1989.

[10] H. Loeliger, Signals sets matched to groups, *IEEE Transactions on Information Theory* 37:1675-1682, 1991.

[11] C. D. Meyer, *Matrix Analysis and Applied Linear Algebra*, Society for Industrial Mathematics (SIAM), Philadelphia PA, USA, 2000.

[12] D. Micciancio, S. Goldwasser, *Complexity of Lattice Problems - A Cryptographic Perspective*, Kluwer Academic Publishers, Norwell, Massachusetts, USA, 2002.

[13] R.M. Siqueira, S.I.R. Costa, Flat tori, lattices and bounds for commutative group codes, *Designs, Codes and Cryptography*, 49:307-321, 2008.

[14] D. Slepian, Group codes for the Gaussian channel, *Bell System Technical Journal*, 47:575-602, 1968.

[15] N.J.A. Sloane, V.A. Vaishampayan, S.I.R. Costa, A note on projecting the cubic lattice. *Discrete Comput. Geom.* 46(3):472-478, 2011.

[16] C. Torezzan, S.I.R. Costa, V.A. Vaishampayan, Spherical codes on torus layers, In *Proceedings of the IEEE International Symposium on Information Theory*, 2009.

[17] C. Torezzan, J.E. Strapasson, S.I.R. Costa, R.M. Siqueira, Optimum commutative group codes, *Des. Codes Cryptogr.*, 74(2):379-394, 2015.

[18] V.A. Vaishampayan, S.I.R. Costa, Curves on a sphere, shift-map dynamics and error control for continuous alphabet sources, *IEEE Transactions on Information Theory* 49:1658-1672, 2003.

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