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A note on two conjectures relating the independence number and spectral radius of the signless Laplacian matrix of a graph

Jorge Alencar[1]
IFSULDEMINAS, Inconfidentes, MG
Leonardo Lima[2]
Departamento de Engenharia de Produção, CEFET-RJ 20271-110, Maracanã, RJ

Abstract. Let G be a simple graph. In this paper, we disprove two conjectures proposed by P. Hansen and C. Lucas in the paper *Bounds and conjectures for the signless Laplacian index of graphs*. We find an infinite class of graphs as a counterexample for two conjectures relating the spectral radius of the signless Laplacian and the independence number of G.

Keywords. conjecture, counterexamples, signless Laplacian matrix, independence number.

1 Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = \{1, \dots, n\}$ and edge set E. Let d_i denote the degree of the vertex $i \in V, i = 1, 2, ..., n$, and $D = D(G) = diag(d_1, d_2, \dots, d_n)$ be the diagonal matrix of the vertex degrees. As usual, we write $Q(G) = D(G) + A(G)$ for the signless Laplacian matrix of a graph G, where $A(G)$ is the well-known $(0, 1)-$matrix, i.e., the adjacency matrix. It is easy to see that $Q(G)$ is symmetric and positive semidefinite. The eigenvalues of the Q matrix can be arranged in non-increasing order by

$$q_1 \geq q_2 \geq \dots \geq q_n \geq 0.$$

The largest eigenvalue of Q, denoted by q_1, is called the spectral radius of Q. A subset $U \subset V$ is an independent vertex set if subgraph induced by U is an empty graph. The independence number of a graph is the largest cardinality of U and is denoted by α.

Hansen and Lucas in [1] established two conjectures relating the eigenvalue q_1 and the independence number α as one can see below.

Conjecture 1 ([1]). *Let G be a connected graph on $n \geq 4$ vertices with signless Laplacian index q_1 and independence number α. Then*

$$4 + \left\lfloor \frac{n}{2} \right\rfloor \leq q_1 + \alpha, \text{ if } n \text{ is odd,} \tag{1}$$

$$2(n-1) \leq q_1 \alpha \tag{2}$$

[1]jorge.alencar@ifsuldeminas.edu.br
[2]llimas@cefet-rj.br

The bound for (1) *is attained by and only by the cycle* C_n *when* n *is odd. Moreover, if* n *is even, then* $q_1 + \alpha$ *is minimal for the graph on* $n \geq 8$ *vertices obtained from two cycles of cardinality* $2\left\lfloor \frac{n}{6} \right\rfloor + 1$ *by linking them by a path. The bound for* (2) *is attained by the complete graph* K_n*, and the odd cycle* C_n *when* n *is odd.*

In this paper, we disprove both inequalities of the Conjecture (1) by defining two classes of graphs which we will call *necklace graph* and *broken necklace graph.*

2 The necklace graph

Let G be a graph obtained from a p-cycle, for $p \geq 3$, by replacing each vertex by a k-clique such that there are two vertices of the clique with degree k in G. In particular, when $p = 2$ we use the same procedure in a multigraph with 2 vertices and 2 edges. Any graph defined in this way will be called a *necklace graph* and we denoted it by $N_{k,p}$. The Figure 1 displays an example of a necklace graph with $k = p = 4$.

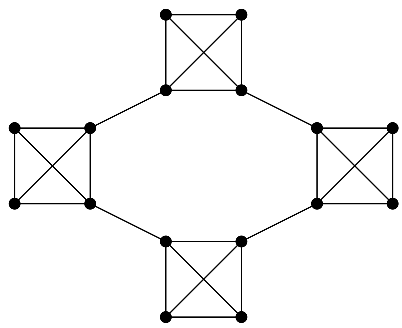

Figure 1: The necklace graph $N_{4,4}$.

Given a partition $\{V_1, \ldots, V_k\}$ of $V(G)$, it is an equitable partition if every vertex in V_i has the same number of neighbours in V_j, for all $i, j \in \{1, \ldots, k\}$. Suppose now that $\mathcal{F} = \{V_1, \ldots, V_k\}$ is an equitable partition of $V(G)$ and that each vertex in V_i has b_{ij} neighbours in V_j ($i, j \in \{1, \ldots, k\}$). Let $D_G(\mathcal{F})$ be the digraph with vertex set $\mathcal{F}$ and b_{ij} arcs from V_i to V_j, and addtional $\sum_{j=1}^{k} b_{ij}$ loops to the vertex V_j, for $j \in \{1, \ldots, k\}$. We call $D_G(\mathcal{F})$ the Q-divisor of G with respect to $\mathcal{F}$. The adjacency matrix obtained from $D_G(\mathcal{F})$ is called the Q-divisor matrix of $\mathcal{F}$, denoted by $A_G(\mathcal{F})$. More results on divisors of graphs can be seen in [2] and [3]. It is known that any eigenvalue of $A_G(\mathcal{F})$ is also a eigenvalue of Q, in particular, the Lemma 2.1 holds.

Lemma 2.1. *Any Q-divisor of a graph G has the Q-index of G as an eigenvalue.*

Next, we obtain the largest $Q-$eigenvalue of the graph $N_{k,p}$.

Proposition 2.1. *The Q-index of $N_{k,p}$ is given by*

$$q_1 = \frac{1}{2}(3k - 2 + \sqrt{(k-2)^2 + 16})$$

Proof. If we define $V_i = \{j \in V : i \equiv j \bmod(k)\}$, for $i = 0, \ldots, k-1$, and $W_1 = V_0$, $W_2 = V_{k-1}$ and $W_3 = \bigcup_{i=1}^{k-2} V_i$, then $\mathcal{F} = \{W_1, W_2, W_3\}$ is an equitable partition of $N_{k,p}$ which generates the following divisor of $N_{k,p}$

$$D_{N_{k,p}}(\mathcal{F}) = \begin{bmatrix} 2k-4 & 1 & 1 \\ k-2 & k & 2 \\ k-2 & 2 & k \end{bmatrix}$$

with spectrum given by

$$\left\{\frac{1}{2}(3k-2-\sqrt{(k-2)^2+16}), k-2, \frac{1}{2}(3k-2+\sqrt{(k-2)^2+16})\right\}.$$

Then, by the Lemma 2.1

$$q_1 = \frac{1}{2}(3k-2+\sqrt{(k-2)^2+16}).$$

□

Each of the subsets V_i, $i = 0, 1, \ldots, k-1$, generates a independent set. Then $\alpha \geq p$.

Proposition 2.2. *For $N_{k,p}$, we have $\alpha = p$.*

Proof. Let S be an independent set such that $|S| = \alpha$. Suppose $\alpha > p$, then, since there are p disjoint cliques with size k, then by pigeonhole principle, at least two elements of S are in a same clique which is an absurd. Thus $\alpha = p$. □

Theorem 2.1. *For $p \geq 5$ or $k \geq 5$, we have $N_{k,p}$ disproves Conjecture 1 equation (1).*

Proof. Suppose that inequality (1) of Conjecture 1 is true for $N_{k,p}$,

$$\alpha q_1(N_{k,p}) \geq 2(n-1).$$

Thus,

$$\begin{aligned} \alpha q_1(N_{k,p}) &\geq 2(n-1) \\ \frac{p}{2}(3k-2+\sqrt{(k-2)^2+16}) &\geq 2(pk-1) \\ 3pk-2p+p\sqrt{(k-2)^2+16} &\geq 4pk-4 \\ p\sqrt{(k-2)^2+16} &\geq pk+2p-4 \\ p^2((k-2)^2+16)-(pk+2p-4)^2 &\geq 0 \\ -8(2+p((k-2)(p-1)-4)) &\geq 0. \end{aligned}$$

Since $(k-2)(p-1) \geq 4$, if $p \geq 5$ or $k > 5$, then

$$\begin{aligned} (k-2)(p-1)-4 &\geq 0 \\ 2+p((k-2)(p-1)-4) &\geq 2 \\ -8(2+p((k-2)(p-1)-4)) &\leq -2. \end{aligned}$$

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which means,

$$-2 \geq -8(2 + p((k-2)(p-1) - 4)) \geq 0$$

what is an absurd. Therefore, $N_{k,p}$ is a counterexample for Conjecture 1 when $p \geq 5$ or $k > 5$. Now, if $k = 5$ then

$$q_1 = q_1(C_{5,p}) = \frac{1}{2}(3k - 2 + \sqrt{(k-2)^2 + 16}) = 9.$$

Thus,

$$\alpha q_1 = 9p \leq 10p - 2 = 2(5p - 1) = 2(n - 1),$$

where equality holds if and only if $p = 2$. If $p > 2$, inequality (1) is not true for $C_{5,p}$. Besides, if $p = 2$, then $C_{5,p}$ contradicts the equality conditions. Therefore, $N_{k,p}$ is a counterexample to Conjecture 1 inequality (1) when $p \geq 5$ or $k \geq 5$. □

Theorem 2.2. *For $p \geq 4$ and $k \geq 3$, we have $N_{k,p}$ disproves Conjecture 1 equation* (2).

Proof. Since $n = pk$, we can rewrite Conjecture 1 equation (2) as follows

$$4 + \left\lfloor \frac{pk}{2} \right\rfloor \leq p + \frac{1}{2}(3k - 2 + \sqrt{(k-2)^2 + 16}).$$

Suppose that the Conjecture 1 equation (2) is true. Thus, is pk is even, then

$$\begin{aligned}
8 + pk &\leq 2p + 3k - 2 + \sqrt{(k-2)^2 + 16} \\
10 + pk - 2p - 3k &\leq \sqrt{(k-2)^2 + 16} \\
(k-2)(p-3) + 4 &\leq \sqrt{(k-2)^2 + 16} \\
((k-2)(p-3) + 4)^2 &\leq (k-2)^2 + 16 \\
(k-2)^2(p-3)^2 + 8(k-2)(p-3) &\leq (k-2)^2 \\
8(k-2)(p-3) &\leq (4-p)(k-2)^2
\end{aligned}$$

So, if $p = 4$, then

$$0 < 8(k-2) \leq 0 \cdot (k-2)^2 = 0,$$

which is an absurd. If $p > 4$, then

$$0 < 8(k-2)(p-3) \leq (4-p)(k-2)^2 < 0,$$

which is an absurd. Now, suppose pk is odd, thus, following the same procedure, we have

$$8(k-2)(p-3) \leq (4-p)(k-2)^2 + 7.$$

Since pk is odd, we have $p > 4$, thus

$$8 \leq 8(k-2)(p-3) \leq (4-p)(k-2)^2 + 7 \leq 7,$$

which is an absurd. Therefore, $N_{k,p}$ disproves inequality (2) of Conjecture 1 when $p \geq 4$ and $k \geq 3$. □

3 The broken necklace graph

Let G be a graph obtained from a p-path, for $p \geq 2$, by replacing each vertex by a k-clique such that if the vertex of the path is an end vertex, then there is only one vertex from the clique with degree k in G, otherwise there are two vertices from the clique with degree k in G. Any graph defined as above will be called a *broken necklace graph* and denoted by $BN_{k,p}$. The Figure 2 displays an example of a necklace graph with $k = p = 4$.

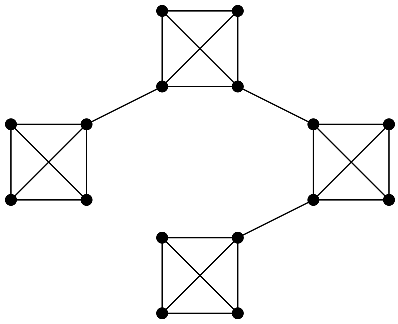

Figure 2: An example of $BN_{4,4}$

Each subset V_i, $i = 0, 1, 2$, yields an independent set and then $\alpha \geq p$.

Proposition 3.1. *For $BN_{k,p}$, we have $\alpha = p$.*

Proof. Let S be an independent set such that $|S| = \alpha$. Suppose $\alpha > p$, then, since there are p disjoint cliques with size k, then by pigeonhole principle, at least two elements of S are in a same clique. Absurd, thus $\alpha = p$. □

Theorem 3.1. *For $p \geq 2$ or $k \geq 3$, we have $BN_{k,p}$ disproves Conjecture 1 equation* (1).

Proof. For $p \geq 5$ or $k \geq 5$ we have

$$\alpha q_1(BN_{k,p}) = \alpha q_1(N_{k,p} - e) < \alpha q_1(N_{k,p}) < 2(n-1)$$

where the last inequality hold by Theorem 2.1. For $5 > p \geq 2$ and $5 > k \geq 3$, verify it computationally. □

Theorem 3.2. *For $p \geq 4$ and $k \geq 3$, we have $BN_{k,p}$ disproves Conjecture 1 equation* (2).

Proof. For $p \geq 4$ and $k \geq 3$,

$$\alpha + q_1(BN_{k,p}) = \alpha + q_1(N_{k,p} - e) < \alpha + q_1(N_{k,p}) < 4 + \left\lfloor \frac{n}{2} \right\rfloor$$

where the last inequality hold by Theorem 2.2. □

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