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An explicit numerical method for random differential equations driven by diffusion-type noises

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In this work we propose a numerical integrator with appealing B-stability properties for the effective integration of Random Ordinary Differential Equations (RDEs) under the influence of Itô-diffusion noise. Basically the introduced integrators are obtained by transforming the RDE to a stochastic differential equation and then adapting the well-known local linearization approach to the special structure of the resulting equation. The introduced method enables to overcome much of the numerical instability that are frequently found when using explicit integrators and is computationally more efficient than stable implicit ones. Results on the convergence and stability of the proposed method are discussed and we also outline some key issues concerning the efficient computational implementation of the corresponding numerical schemes.

Keywords. random differential equations, numerical approximation, B-stability, convergence, stochastic differential equations

1 Introduction

Noise plays an important role in modelling dissimilar phenomena. In particular, the mathematical modelling of physical systems and several real-world problems leads to differential equations containing some inherent randomness due to uncertainties. A differential system can involve uncertainly in different ways. Particularly, during the last decades, in different areas including engineering, biology and social sciences, deterministic differential models have been replaced by Random Differential Equations (RDEs) of the form (see e.g., [8])

$$\mathbf{x}^{'}(t) = f(t, \mathbf{x}(t), \mathbf{y}(t)),\ t \in [t_0, T], \tag{1}$$

which are pathwise Ordinary Differential Equations (ODEs) containing a multidimensional stochastic process $\mathbf{y}(t)$ in their vector field function, where $\mathbf{y}(t) \in \mathbb{R}^n$ designates a random input process uncoupled with the solution $\mathbf{x}(t) \in \mathbb{R}^d$. The study of this kind of equations is very important in modern applied mathematics and in fact nowadays RDEs are used in a wide range of applications, see e.g. [1], [13], [12], [11], [15], [16], [17].

Just as in the deterministic case, closed-form expressions for the solution of RDEs are often unobtainable, and so the construction of approximation methods for the treatment and simulation of RDEs has become an important need.

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In this paper we are concerned with the approximation of RDEs for which the random process $\mathbf{y}(t)$ is an Ito-diffusion driven by additive noise. That is, $\mathbf{y}(t)$ satisfies a stochastic differential equation (SDE) of the form

$$d\mathbf{y}(t) = a\left(t, \mathbf{y}(t)\right) dt + \sum_{j=1}^{m} b_j\left(t, \mathbf{y}(t)\right) dw_t^j, \qquad t \in [t_0, T], \tag{2}$$

where $(w_t^1, \ldots, w_t^m)$ denotes an m-dimensional standard Wiener process. Here, $\mathbf{y}(t) \in \mathbb{R}^n$, and a, b_j are smooth enough nonlinear functions.

At a first glance one could think that some of the existing numerical schemes for ODEs can be used pathwise for RDEs, but the driving stochastic process $\mathbf{y}(t)$ has at most Hölder continuous sample paths, so the vector field in (1) is at most Hölder continuous with respect the time variable, so numerical schemes for ODEs when applied to RDEs are not convergent or rarely attain their traditional order.

Similarly to the deterministic scenario, there exists a variety of important issues in designing practical numerical integrators for RDEs. In particular, in the stochastic scenario stability and computational efficiency of the numerical schemes are the more important and desirable properties. Taking all this into consideration, some numerical integrators have been proposed in literature e.g., [4], [3], [7], [9], [2]. However, most of these methods or are of an implicit nature (involving the numerical solution of a system of nonlinear algebraic equations at each integration step, that typically increase the computational effort of these numerical integrators) or are explicit integrators, having the appealing feature of retaining the standard order of convergence of the classical deterministic schemes, but at the expense of high computational cost and low stability.

The aim of the present work is to introduce an explicit and B-stable numerical integrator for (1)-(2). For this we adapt the local linearization approach (see [5]) -which has been successfully applied in the framework of deterministic and stochastic differential equations- to the special structure of the resulting equation (1)-(2) after rewriting it as a full SDE. Consequently by exploiting this special structure of this SDE our approach will provide a method able to overcome much of the numerical instability that are frequently found when using explicit integrators and being computationally more efficient than stable implicit ones.

The paper is organized as follows. After this introduction, section 2 presents the deduction of the proposed method and the convergence and stability is considered. In section 3 are given details on the effective implementation of the exponential involved in the proposed method.

2 The proposed integrator

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and $(\mathcal{F}_t)_{t\geq 0}$ be an increasing right continuous family of complete sub σ-algebras of $\mathcal{F}$. Consider the d-dimensional RDE (1) where $\mathbf{y}(t)$ is a $\mathcal{F}_t$-adapted finite continuous Itô-diffussion processes solution of the SDE (2).

Interpreting the RDEs (1) as a $(d+n)-$dimensional SDEs we obtain the augmented system

$$d\begin{pmatrix}\mathbf{x}(t)\\ \mathbf{y}(t)\end{pmatrix}=\begin{pmatrix}f(t,\mathbf{x}(t),\mathbf{y}(t))\\ a\,(t,\mathbf{y}(t))\end{pmatrix}dt+\sum_{j=1}^{m}\begin{pmatrix}\mathbf{0}\\ b_j\,(t,\mathbf{y}(t))\end{pmatrix}dw_t^j, \tag{3}$$

The idea to construct our method is to use the partially-uncoupled structure of this equation to adapt conveniently the local linearization technique from [5].

Let $(t)_h=\{t_n : n=0,1,\ldots,N\}$ be a partition of the time interval $[t_0,T]$ with, for simplicity, equidistant stepsize $h<1$, i.e., defined as a sequence of times $t_0<t_1<\ldots<t_N=T$ such that $t_n=t_0+nh$, for $n=0,1,\ldots,N$. Starting from the initial value x_0, the approximations $\{x_i\}$ to $\{\mathbf{x}\,(t_i)\}$, $(i=1,2,\ldots,N)$ are obtained recursively as follows.

For each time interval $[t_n,t_{n+1}]$ we use the Itô-formula to obtain the linearization

$$\begin{pmatrix}f(t,\mathbf{x}(t),\mathbf{y}(t))\\ a\,(t,\mathbf{y}(t))\end{pmatrix}\approx\begin{pmatrix}f_n\\ a_n\end{pmatrix}+J_n\,\begin{pmatrix}x(t)-x_n\\ y(t)-y_n\end{pmatrix}+q_n\,(t-t_n),$$

with

$$J_n=\begin{pmatrix}[f_x]_n & [f_y]_n\\ \mathbf{0} & [a_y]_n\end{pmatrix}=\begin{pmatrix}\left[\frac{\partial f}{\partial \mathbf{x}}\right]_{(t_n,x_n,y_n)} & \left[\frac{\partial f}{\partial \mathbf{y}}\right]_{(t_n,x_n,y_n)}\\ \mathbf{0} & \left[\frac{\partial a}{\partial \mathbf{y}}\right]_{(t_n,y_n)}\end{pmatrix},$$

$$q_n=\begin{pmatrix}[q^1]_n\\ [q^2]_n\end{pmatrix}=\begin{pmatrix}\frac{\partial f}{\partial t}(t_n,x_n,y_n)\\ \left[\frac{\partial a}{\partial t}+\frac{1}{2}\sum_{j=1}^{m}(\mathbf{I}\otimes b_j^{\intercal})\;a_{xx}\;b_j\right]_{(t_n,y_n)}\end{pmatrix},$$

$$f_n=f(t_n,x_n,y_n),\;\; a_n=a\,(t_n,y_n)\,.$$

Now the solution $\mathbf{x}(t_{n+1})$ of (1) in $\mathbf{\Lambda}_n$ could be approximated by the solution of the linear SDE

$$d\begin{pmatrix}\mathbf{x}(t)\\ \mathbf{y}(t)\end{pmatrix}=\left(J_n\,\begin{pmatrix}x(t)-x_n\\ y(t)-y_n\end{pmatrix}+q_n\,(t-t_n)+\begin{pmatrix}f_n\\ a_n\end{pmatrix}\right)dt+\sum_{j=1}^{m}\begin{pmatrix}\mathbf{0}\\ b_j\,(t,y_n)\end{pmatrix}dw_t^j,$$

$$\begin{pmatrix}\mathbf{x}(t)\\ \mathbf{y}(t)\end{pmatrix}=\begin{pmatrix}x_n\\ y_n\end{pmatrix}.$$

However, taking advantage of the special structure of the equation above (note that $\mathbf{y}(t)$ is uncoupled with $\mathbf{x}(t)$), our proposal to approximate $\mathbf{x}(t_{n+1})$ is to solve first the equation for $\mathbf{y}(t)$ and then to consider the linear nonhomogeneous equation for $\mathbf{x}(t)$, by using y_n and also the previously computed approximation to $\mathbf{y}(t_{n+1})$.

In this way

$$y_{n+1}=y_n+\mathbf{M}_{14}+\left(\mathbf{M}_{12}\mathbf{M}_{11}^{\top}\right)^{\frac{1}{2}}\xi_n,$$

where

$$\mathbf{M}=e^{\mathbf{C}_n h},$$
$$\mathbf{C}_n = \begin{bmatrix} [a_y]_n & [bb^\top]_n & [q^1]_n & a_n \\ \mathbf{0} & -[a_y]_n^\top & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 0 & 1 \\ \mathbf{0} & \mathbf{0} & 0 & 0 \end{bmatrix},$$

with $b = (b_1, \ldots, b_k)^\top$ and ξ_n a sequence of $k-$dimensional *i.i.d.* normal random vectors.

By using this computed value y_{n+1} and solving the linear random (but not stochastic) equation for $\mathbf{x}(t)$ we finally obtain

$$x_{n+1} = x_n + \mathbf{L}e^{\mathbf{D}_n h}\mathbf{r}, \tag{4}$$

where

$$\mathbf{D}_n = \begin{bmatrix} [f_x]_n & \left(\frac{1}{h}[f_y]_n\left(\mathbf{M}_{14} + \left(\mathbf{M}_{12}\mathbf{M}_{11}^\top\right)^{\frac{1}{2}}\xi_n\right) + [q^1]_n\right) & f_n \\ \mathbf{0} & 0 & 1 \\ \mathbf{0} & 0 & 0 \end{bmatrix} \in \mathbb{R}^{(d+2)\times(d+2)},$$
$$\mathbf{L} = [\mathbf{I}_d \;\; \mathbf{0}_{d\times 2}], \;\; \mathbf{r} = \begin{bmatrix} \mathbf{0}_{(d+1)\times 1} \\ 1 \end{bmatrix},$$

We refer to [5] for details about how this representation can be obtained.

2.1 Convergence and stability

2.1.1 Pathwise convergence

In this section main results concerning the trajectory-wise convergence and B-stability of the methods is considered.

We have the following theorems:

Theorem: Let's suppose that there exist almost surely a finite stochastic processes $L(t)$ such that the Lipschitz condition

$$\|f(t,u,y) - f(t,v,y)\| \leq L(t)\|u - v\|,$$

is satisfied. Also suppose that a, b are smooth enough. Then the numerical integrator (4) is almost surely globally convergent and we have that with probability one

$$\sup_n \|\mathbf{x}(t_n) - x_n\| = O(h),$$

and for $b = b(t)$

$$\sup_n \|\mathbf{x}(t_n) - x_n\| = O(h^{1.5}).$$

2.1.2 B-stability

Let us consider a RDE such that the vector field is dissipative, that is

$$\langle f(t, x_1, y) - f(t, x_2, y), x_1 - x_2 \rangle \leq K \|x_1 - x_2\|^2 ,$$

with $K < 0$.

Then it implies that any solution converges pathwise to a unique stationary solution of the equation.

We are interested in numerical methods reproducing this behavior of the continuos one. In such a case the method is termed B-stable.

We have the following important theorem

Theorem: The numerical integrator (4) is B-stable. In fact, for any two numerical map (4) x_n and z_n we have

$$\|x_{n+1} - z_{n+1}\| \leq \|x_n - z_n\| .$$

3 Implementation issues

The numerical implementation of $\{x_n\}$ is reduced to the use of a algorithm to compute exponential of matrices. In particular, those algorithms based on the rational (p, q)-Padé approximation ($p \leq q \leq p+2$) combined with the "scaling and squaring" strategy provide stable approximations to the matrix exponential. Nowadays, professional mathematical software, such as MATLAB, provide efficient codes for implementing a number of such algorithms. (see [6], [14]) in such a way that the computational saving achieved are very significant.

We first summarize the existing Padé algorithm with "scaling and squaring" strategy on which we based our computer implementation of the method.

3.0.3 The Padé algorithm for computing the matrix exponential

The (p, q) rational Padé approximation to $e^{\mathbf{C}}$ is defined by

$$\mathbf{P}_{p,q}(\mathbf{C}) = [\mathbf{D}_{p,q}(\mathbf{C})]^{-1}\mathbf{N}_{p,q}(\mathbf{C}),$$

where

$$\mathbf{N}_{p,q}(\mathbf{C}) = \sum_{j=0}^{p} \frac{(p+q-j)!p!}{(p+q)!j!(p-j)!}\mathbf{C}^j,$$

and

$$\mathbf{D}_{p,q}(\mathbf{C}) = \sum_{j=0}^{q} \frac{(p+q-j)!q!}{(p+q)!j!(q-j)!}(-\mathbf{C})^j.$$

Diagonal approximation (that is, $p = q$) are preferred, since $\mathbf{P}_{p,q}$ with $p > q$ ($p < q$) is less accurate than $\mathbf{P}_{p,p}$ ($\mathbf{P}_{q,q}$), and $\mathbf{P}_{p,p}$ ($\mathbf{P}_{q,q}$) can be evaluated at the same cost. From now on, we denote $\mathbf{D}_{q,q}(\mathbf{C})$, $\mathbf{N}_{q,q}(\mathbf{C})$, $\mathbf{P}_{q,q}(\mathbf{C})$ by $\mathbf{D}_q(\mathbf{C})$, $\mathbf{N}_q(\mathbf{C})$, $\mathbf{P}_q(\mathbf{C})$ respectively.

$e^{\mathbf{C}}$ can be well approximated by Padé only near the origin, that is, for small $\|\mathbf{C}\|$. For this reason $e^{\mathbf{C}}$ is approximated by $\left(\mathbf{P}_q(\frac{\mathbf{C}}{m})\right)^m$ where m is the minimum integer such that $\left\|\frac{\mathbf{C}}{m}\right\| < \frac{1}{2}$. In order to reduce the number of matrix multiplications, the idea is to choose m to be a power of two. Then $\left(\mathbf{P}_q(\frac{\mathbf{C}}{m})\right)^m$ can be efficiently computed by repeated squaring.

The Padé algorithm with scaling-squaring strategy for computing $e^{\mathbf{C}}$ can be described as follows.

1. Determine the minimum integer k such that $\left\|\frac{\mathbf{C}}{2^k}\right\| < \frac{1}{2}$
2. Compute $\mathbf{N}_q(\frac{\mathbf{C}}{2^k})$ and $\mathbf{P}_q(\frac{\mathbf{C}}{2^k})$
3. Compute $\mathbf{P}_q(\frac{\mathbf{C}}{2^k}) = [\mathbf{D}_q(\frac{\mathbf{C}}{2^k})]^{-1}\mathbf{N}_q(\frac{\mathbf{C}}{2^k})$, by solving the system $\mathbf{D}_q(\frac{\mathbf{C}}{2^k})\mathbf{P}_q(\frac{\mathbf{C}}{2^k}) = \mathbf{N}_q(\frac{\mathbf{C}}{2^k})$ (using, for instance, a suitable Gaussian elimination)
4. Compute $[\mathbf{P}_q(\frac{\mathbf{C}}{2^k})]^{2^k}$ by squaring $\mathbf{P}_q(\frac{\mathbf{C}}{2^k})$ k times

4 Conclusions

In this work we introduce an explicit numerical integrator for the computer simulation of the RDEs driven by Itô-diffusion processes. We analyzed the convergence and the B-stability of the method. Remarkably the method is explicit and is fully stable. This suggest the potential applicability of this integrator for stiff RDEs, which points out to future work.

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