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Simulation of coupled Stokes-Darcy equations using the combination of balanced $\mathbf{H}(\mathrm{div})$-L^2 approximation spaces

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Abstract. The performance of a strongly conservative finite element formulation for coupled Stokes-Darcy problems is discussed. The same pair of balanced $\mathbf{H}(\mathrm{div})$-L^2 approximation spaces is used for flow an pressure variables, in both sides. For Darcy's flux, $\mathbf{H}(\mathrm{div})$-conforming spaces is the natural context. In the Stokes side, the discontinuity of tangential flux components is treated by a penalization term, as in usual Discontinuous Galerkin methods. For incompressible fluids, this method naturally gives exact divergence-free velocity fields, a property that few schemes can achieve. Furthermore, the coupling Stoke-Darcy interface conditions is naturally enforced. The method is implemented using an object-oriented computational environment, and a test problem is simulated to illustrate numerical approximation properties, verifying errors and rates of convergence.

Keywords. Mixed Finite Elements Methods. Discontinuous Galerkin. Stokes flow, Coupled Stokes-Darcy flow. Divergence-free Simulation.

1 Introduction

In this paper, the interest is on finite element approximations of coupled Stokes–Darcy problem, which appear in considerable number of applications [1]. For instance, this is the case of modeling the interaction of flows in wells and reservoirs, in fractured porous media, commonly found in Petroleum Engineering, and between surface (rivers) and groundwater (aquifer). Free flow channel confined by porous walls is also a feature of many of the natural and industrial settings [2]. Other application is for self-compacting concrete flow around reinforcing bars, where the reinforced bar domain is represented by a Darcy's law obtained by homogenization, while a Stokes flow is considered elsewhere [3].

There are several works in the literature dealing with coupled Stokes-Darcy problems, and we refer to [4–6] for extensive bibliography review on this topic. Recently, it has

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regained relevance and is attracting new attention of numerical analysts, and improved numerical methods have been proposed [7].

For coupled Stokes-Darcy problems, we adopt $\mathbf{H}(\text{div})$-conforming flux approximations on both sides, a method that has been analyzed in [5, 8]. On Stokes's part, the tangential discontinuity of vector functions in the $\mathbf{H}(\text{div})$ subspaces is treated by a penalization term, as in usual Discontinuous Galerkin methods. For Darcy's flow, this is a natural context, and a classic mixed formulation is applied [9]. In order to fit the fluids between the two domains, experimental conditions presented in [10] shall be considered, relating tangential component of fluid velocity and shear stress on Stokes-Darcy interface.

As emphasized in [8], the most important property of this method is its capability of solving strongly the divergence-free equation, a task that few schemes can accomplish. Furthermore, the same pair of approximation spaces for velocity and pressure representations can be used in both flow regions, a characteristic not shown by most methods designed for the simulation of coupled Stokes-Darcy problems. Another advantage is that the Beavers–Joseph–Saffman condition is easier to enforce, since the bi-linear form only involves the tangential component of the velocity over the Stokes-Darcy interface.

A object oriented programming environment, called NeoPZ [11], was used for the computational implementation of all the presented methods. Developed at LabMeC (Laboratory of Computational Mechanics), at the State University of Campinas, Brazil, it can be freely downloaded from http://github.com/labmec/neopz.

The organization of the present paper starts by setting the main notation in Section 2. Next, the coupled Stokes-Darcy problem is considered in Section 3, and numerical results are shown. In all the simulations, Raviart-Thomas [12] space configuration based on quadrilateral elements are used. Finally, in Section 4, we present the conclusions.

2 Notation

Let $\Omega \subset \mathbb{R}^2$ be an open polygonal domain with border $\partial\Omega$, and unit normal $\vec{n}$ exterior to Ω. Shape-regular partitions $\mathcal{T} = \{\Omega_e, e = 1, \cdots, n_{el}\}$ of Ω, formed by affine quadrilateral elements, shall be considered. The set Γ formed by all element edges E is called the mesh skeleton, and $\Gamma_{int} = \{E \in \Gamma : E \subset \Omega\}$ denotes the set of internal edges. To each interior edge E, once and for all, a unit normal vector $\vec{n}_E$ and a tangent vector $\vec{\tau}_E$ are associated so that $\{\vec{n}_E, \vec{\tau}_E\}$ form a right-hand coordinate system. If E is a boundary edge, then $\vec{n}_E = \vec{n}$ is the unit normal exterior to Ω. Over interfaces $E \in \Gamma_{int}$ between two elements, Ω_1 and Ω_2, jump and average operators of a function v are formally defined as

$$[\![v]\!]_E = v_1|_E - v_2|_E, \quad \langle v \rangle_E = \frac{1}{2}\left(v_1|_E + v_2|_E\right),$$

where $v_i = v|_{\Omega_i}$. For boundary edges $E \subset \partial\Omega$, jump and average are function traces over E.

Local approximations shall be defined in terms of one of the following polynomial spaces:

- $\mathbb{Q}_k$: scalar polynomials of maximum degree k in each variable.

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- $\mathbb{Q}_{m,n}$: scalar polynomials of maximum degree m in x, and n in y.

Vector spaces $\overrightarrow{V}$ are piece-wise defined over the elements $\Omega_e \in \mathcal{T}$ in terms of local polynomial approximations $\overrightarrow{V}_e$. It is here defined:

$$\overrightarrow{V} \quad = \quad \{\vec{\varphi} \in \mathbf{H}(div, \Omega);\ \vec{\varphi}|_{\Omega_e} \in \overrightarrow{V}_e\}.$$

It is clear that for $E \in \Gamma_{int}$,

$$[\![\vec{\varphi}]\!]_E = \quad [\![(\vec{\varphi} \cdot \vec{\tau}_e)]\!]_E \vec{\tau}_E \quad \text{if } \vec{\varphi} \in \overrightarrow{V},$$

since $[\![\vec{\varphi} \cdot \vec{n}]\!]_E = 0$ for $\mathbf{H}$(div)-conforming vector fields.

The scalar approximation spaces for the pressure are also piece-wise defined in terms of local polynomial spaces Ψ_e. Globally, the spaces may be continuous or discontinuous

$$\Psi \quad = \quad \{\varphi \in L^2_0(\Omega);\ \varphi|_{\Omega_e} \in \Psi_e\},$$

where $L^2_0(\Omega) = \{\varphi \in L^2(\Omega);\ \int_\Omega \varphi \, d\Omega = 0\}$.

3 Coupled Stokes-Darcy model

We present in this section a finite element formulation for the coupled model considering a Darcy's flow in a region $\Omega_{\mathcal{D}}$ and a Stokes flow in a region $\Omega_{\mathcal{S}}$. Both regions form the computational domain $\Omega = \Omega_{\mathcal{D}} \cup \Omega_{\mathcal{S}}$, and are assumed to be polygonal, sharing the interface $\Gamma_{\mathcal{SD}} = \partial\Omega_{\mathcal{D}} \cap \partial\Omega_{\mathcal{S}}$. Let $\vec{n}_{\mathcal{SD}}$ be unit normal vectors to the edges in $\Gamma_{\mathcal{SD}}$, pointing from $\Omega_{\mathcal{S}}$ to $\Omega_{\mathcal{D}}$, and let $\vec{\tau}_{\mathcal{SD}}$ be the associated positive oriented tangent vectors. For convenience, restrictions of functions to each of the flow domains are denoted by $v_{\mathcal{S}} = v|_{\Omega_{\mathcal{S}}}$, and $v_{\mathcal{D}} = v|_{\Omega_{\mathcal{S}}}$.

The coupled Stokes-Darcy model problem consists in finding $\vec{u}$ and p such that:

$$\begin{aligned} -\nabla \cdot \mathbb{T}(\vec{u}, p) &= \vec{f} && \text{in} \quad \Omega_{\mathcal{S}}, \\ \mu\mathbb{K}^{-1}\,\vec{u} + \nabla p &= 0 && \text{em} \quad \Omega_{\mathcal{D}}, \\ \nabla.\,\vec{u} &= f && \text{in} \quad \Omega, \end{aligned}$$

where $\mathbb{K}$ is the permeability tensor, $\vec{f} \in [L^2(\Omega_{\mathcal{S}})]^2$, and $f \in L^2(\Omega)$, with $f|_{\Omega_{\mathcal{S}}} = 0$. The boundary conditions for this coupled problem are:

$$\vec{u} = 0 \quad \text{in} \quad \partial\Omega_{\mathcal{S}} \setminus \Gamma_{\mathcal{SD}}, \qquad \vec{u}\,.\,\vec{n} = 0 \quad \text{in} \quad \partial\Omega_{\mathcal{D}} \setminus \Gamma_{\mathcal{SD}}.$$

Furthermore, the following conditions should be enforced at the interface $\Gamma_{\mathcal{SD}}$:

- Flux continuity: $\vec{u}_{\mathcal{S}} \cdot \vec{n}_{\mathcal{SD}} = \vec{u}_{\mathcal{D}} \cdot \vec{n}$.
- Balance of normal forces: $p_{\mathcal{S}} - 2\mu\,[D(\vec{u}_{\mathcal{S}})\,\vec{n}_{\mathcal{SD}}] \cdot \vec{n}_{\mathcal{SD}} = p_{\mathcal{D}}$.
- Beavers-Joseph-Saffman (BJS) condition [10]: $\vec{u}_{\mathcal{S}} \cdot \vec{\tau}_{\mathcal{SD}} = -2G\,[D(\vec{u}_{\mathcal{S}})\,\vec{n}_{\mathcal{SD}}] \cdot \vec{n}_{\mathcal{SD}}$, where $G > 0$ is an empirical coefficient.

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3.1 Weak formulations for the Stokes-Darcy problem

The main goal here is to show how the well known capacity of divergence-conforming spaces for flux representation in mixed methods for Darcy's flows can be combined with a Stokes flow scheme, in order to design different stable and accurate weak formulations for the coupled Stokes-Darcy problem.

Let $\mathcal{T}_\mathcal{S}$ and $\mathcal{T}_\mathcal{D}$ be shape-regular partitions of $\Omega_\mathcal{S}$ and $\Omega_\mathcal{D}$, respectively, and assume they match along the interface $\Gamma_{\mathcal{SD}}$. This means that $\mathcal{T} = \mathcal{T}_\mathcal{S} \cup \mathcal{T}_\mathcal{D}$ form a partition of $\Omega_\mathcal{S} \cup \Gamma_{\mathcal{SD}} \cup \Omega_\mathcal{S}$. Given pressure and velocity approximation spaces Ψ and $\overrightarrow{V}$ based on the partition $\mathcal{T}$, the mixed formulation searches for a pair of functions $\{\vec{u}, p\} \in \overrightarrow{V} \times \Psi$, such that, for $\forall \{\vec{\varphi}, \varphi\} \in \overrightarrow{V} \times \Psi$ we have:

$$
\begin{aligned}
a_{\mathcal{SD}}(\vec{u}, \vec{\varphi}) + b_{\mathcal{SD}}(\vec{\varphi}, p) &= \int_{\Omega_\mathcal{S}} \vec{f} \cdot \vec{\varphi}_\mathcal{S} d\Omega_\mathcal{S} + (BC), && (1)\\
b_{\mathcal{SD}}(\vec{u}, \varphi) &= -\int_{\Omega_\mathcal{D}} f \; \varphi_\mathcal{D} d\Omega_\mathcal{D}. && (2)
\end{aligned}
$$

The bi-linear forms for the coupled problem can expressed as

$$
\begin{aligned}
a_{\mathcal{SD}}(\vec{u}, \vec{\varphi}) &= a_\mathcal{S}(\vec{u}_\mathcal{S}, \vec{\varphi}_\mathcal{S}) + \frac{\mu}{G} \int_{\Gamma_{\mathcal{SD}}} (\vec{u}_\mathcal{S} \cdot \vec{\tau}_{\mathcal{SD}}) \; (\vec{\varphi}_\mathcal{S} \cdot \vec{\tau}_{\mathcal{SD}}) \; ds + \int_{\Omega_\mathcal{D}} \vec{u}_\mathcal{D} \cdot \vec{\varphi}_\mathcal{D} \; d\Omega_\mathcal{D}, && (3)\\
b_{\mathcal{SD}}(\vec{u}, \varphi) &= -b_\mathcal{S}(\vec{u}_\mathcal{S}, \varphi_\mathcal{S}) - \int_{\Omega_\mathcal{D}} \nabla \cdot \vec{u}_\mathcal{D} \; \varphi_\mathcal{D} \; d\Omega_\mathcal{D}, && (4)
\end{aligned}
$$

$$
\begin{aligned}
a_\mathcal{S}(\vec{u}, \vec{\varphi}) &= 2\mu \sum_{e=1}^{nel} \left(\int_{\Omega_e} D(\vec{\varphi}) \cdot D(\vec{u}) \, d\Omega_e \right) - 2\mu \sum_{E \in \Gamma_\mathcal{S}} \left\{ \int_E \langle D(\vec{u}) \vec{n}_E \cdot \vec{\tau}_E \rangle [\![\vec{\varphi} \cdot \vec{\tau}_E]\!] \, ds \right.\\
&\quad - \left. \beta \int_E \langle D(\vec{\varphi}) \vec{n}_E \cdot \vec{\tau}_E \rangle \cdot [\![\vec{u} \cdot \vec{\tau}_E]\!] \, ds \right\} + \sum_{E \in \Gamma_\mathcal{S}} \frac{\gamma_E}{|E|} \int_E [\![\vec{u} \cdot \vec{\tau}_E]\!] \cdot [\![\vec{\varphi} \cdot \vec{\tau}_E]\!] \, ds, && (5)\\
b_\mathcal{S}(\vec{u}, \varphi) &= -\int_{\Omega_\mathcal{S}} \nabla \cdot \vec{u} \, \varphi \, d\Omega_\mathcal{S}. && (6)
\end{aligned}
$$

In relation to $a_\mathcal{S}(\vec{u}_\mathcal{S}, \vec{\varphi}_\mathcal{S})$ and $b_\mathcal{S}(\vec{u}_\mathcal{S}, \vec{\varphi}_\mathcal{S})$, the integrals are restricted to the elements $\Omega_e \in \mathcal{T}_\mathcal{S}$ and to the edges in the skeleton $\Gamma_\mathcal{S}$ that are not included in the interface $\Gamma_{\mathcal{SD}}$. The last term in equation (5) is the penalization required to treat the tangential discontinuity of the Stokes velocity, where $\gamma_E = \gamma_0 k^2$, k being the polynomial order chosen for the approximation of the velocity. For a symmetric formulation, the parameter $\beta = -1$.

The next example verifies the implementation of this methodology of this scheme applied to a Stokes-Darcy problem with known exact solution.

Verification test

The following problem was proposed by [13]. The region is a rectangle where the Stokes domain is $\Omega_\mathcal{S} = (0.0, \pi) \times ([0.0, 1.0)$, and the Darcy domain is $\Omega_\mathcal{D} = (0.0, \pi) \times$

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$(-1.0, 0.0)$, with interface $\Gamma_{\mathcal{SD}} = \{0 < x < \pi, y = 0\}$. Viscosity coefficient $\mu = 1$, constant permeability $\mathbb{K} = \mathbb{I}$, and coefficient for the BJS interface condition $\mu^* = 1$ are adopted. For this problem, the exact velocity and pressure fields are:

$$\vec{u}_{\mathcal{S}} = \begin{pmatrix} v'(y)\cos x \\ v(y)\sin x \end{pmatrix}, \quad p_{\mathcal{S}} = \sin x \sin y, \quad \text{where} \quad v(y) = \frac{1}{\pi^2}\sin^2(\pi y) - 2,$$
$$\vec{u}_D = \begin{pmatrix} (e^{-y} - e^{y})\cos x \\ -(e^{-y} + e^{y})\sin x \end{pmatrix}, \quad p_D = (-e^{-y} + e^{y})\sin x.$$

The forcing functions g and $\vec{f}$ are obtained from these exact solutions.

Approximations $\vec{u}_h \in \overrightarrow{V}_h$, $p_h \in \Psi_h$ are computed using uniform rectangular meshes $\mathcal{T}_h$ with mesh sizes $h_{xi} = \pi/N$, and $h_{y_i} = 2/N$, for $N = 2^j$, $j = 2, \cdots, 6$, and $\mathbf{H}(\mathrm{div}) - L^2$ balanced spaces $\overrightarrow{V}\ \Psi$ and 3. For this verification test, symmetric formulation with balanced Raviart-Thomas space configuration $\overrightarrow{\mathbb{Q}}^{\mathrm{div}}_{RT(k)}\ \mathbb{Q}^{\mathrm{d}}_k$, for which $\overrightarrow{V}_e = \mathbb{Q}_{k+1,k} \times \mathbb{Q}_{k,k+1}$ and $\Psi_e = \mathbb{Q}_k$. The penalization parameter $\gamma_0 = 12$ is considered on the Stokes part. Error analysis for the application of these spaces in Darcy's problems with smooth solutions gives L^2-rates of order $k+1$ for velocity and pressure [12]. For Stokes flows, the predicted orders are $k+1$ for velocity, and k for pressure [14].

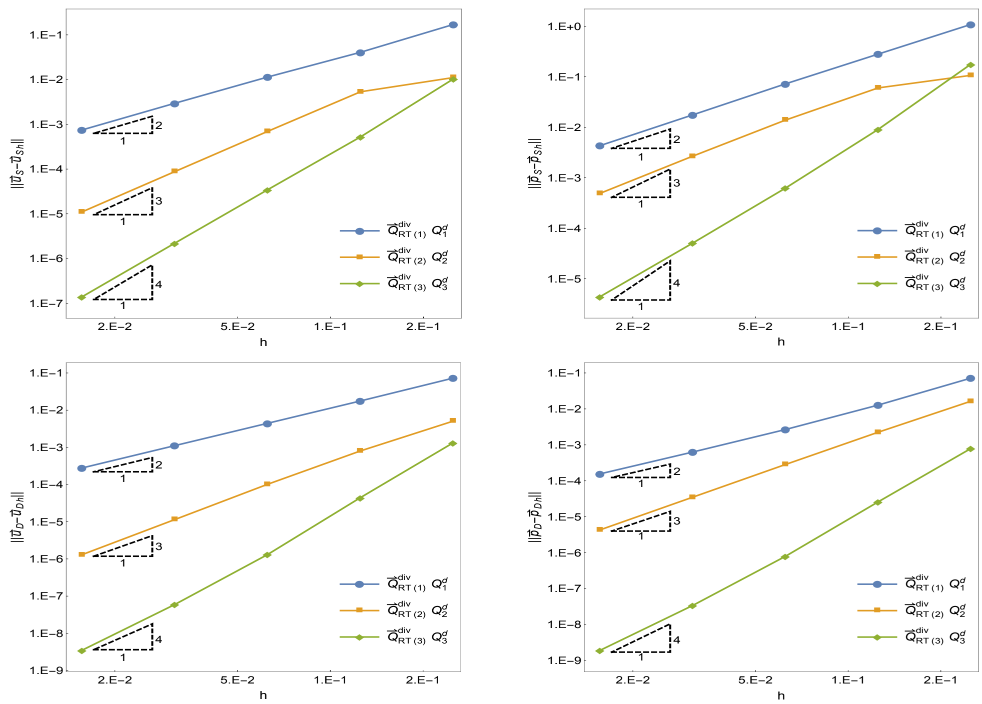

Figure 1: Stokes-Darcy problem: convergence history for $\vec{u}$ (left side), p (right side), on $\Omega_{\mathcal{S}}$ (top side) and on $\Omega_{\mathcal{D}}$ (bottom side), using spaces $\overrightarrow{\mathbb{Q}}^{\mathrm{div}}_{RT(k)}\ \mathbb{Q}^{\mathrm{d}}_k$, $k = 1, 2$ and 3, with symmetric formulation on $\Omega_{\mathcal{S}}$.

The results are represented graphically in Fig. 1, where the predicted errors in Darcy's

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region are achieved. The results for the Stokes flow are also consistent, corresponding to the predicted order $k+1$ for velocity, and rates close to order $k+1$ for pressure, which is higher than the predicted order k.

4 Conclusions

The special interest on the method using balanced $\mathbf{H}(\text{div}) - L^2$ space configurations for the simulation of coupled Stokes-Darcy problem comes mainly from its important capability of solving strongly the divergence-free equation. Furthermore, for coupled flows, the same pair of approximation spaces can be used in both Stokes and Darcy's regions, a convenient characteristic not shown by most methods designed for such problems.

The application of $\mathbf{H}(\text{div})$-conforming spaces revealed to be an efficient form of approximating velocity fields in coupled Stokes-Darcy equations, leading to optimal convergence rates, even when velocity and pressure spaces are defined by polynomials of the same degree, as in the schemes using $\overrightarrow{\mathbb{Q}}^{\mathrm{d}}_{RT(k)}\ \mathbb{Q}^{\mathrm{d}}_k$ and $\overrightarrow{\mathbb{Q}}^{\mathrm{div}}_{RT(k)}\ \mathbb{Q}^{\mathrm{d}}_k$ space configurations based on quadrilateral meshes. Furthermore, the former one has the additional advantage of being strongly conservative, and to allow natural enforcement of Beavers-Joseph-Saffman interface condition in coupled flows, since the bi-linear form only involves the tangential component of the velocity over the Stokes-Darcy interface.

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