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A Comparative Study between Mexican and Golden Hat Wavelets

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Abstract. In this work, a comparative analysis between Mexican Hat and Golden Hat wavelets is presented. The latter wavelet belongs to a new family of functions generated from Fibonacci coefficient polynomials. Although these wavelets have a very similar waveform, they have several distinct characteristics in time and the frequency domains. These distinctions are explored here in the scale-space.

Keywords. Wavelets, Mexican Hat, Fibonacci-coefficient polynomials, Golden Hat.

1 Introduction

Wavelets are analysis functions used in the Wavelet Transform (WT), which is a mathematical tool that splits signals into different frequency bands, which one corresponding to a specific scale [2,7,8]. The wavelets can be complex or real-valued functions. The complex wavelets are generally used to measure the time evolution of frequency transients, in contrast, real wavelets are often used to detect sharp signal transitions [7]. In a general sense, this paper focuses on real wavelets.

The WT coefficients are directly affected by the wavelet function. Thus, the wavelet choice must be made carefully, and it is done directly by the characteristics of the signal to be analyzed [8]. There exist many different types of wavelet functions which can be chosen. A frequently wavelet choice in visual analysis is the second Gaussian derivative, known as *Mexican Hat* [2]. Another possible wavelet choice is the *Golden Hat*. Recenttly presented in [4], the Golden Hat is a member of a wavelet family named *Golden Wavelets*.

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These wavelets are generated from a particular class of polynomials, called by Fibonacci-coefficient polynomials (FCPs), see [3] for more details on these polynomials. However, the first relation between wavelets and FCPs was studied in [6], where a new wavelet function constructed by those polynomials is presented.

The Golden Hat waveform is very similar to the classical Mexican Hat wavelet, which can see in [4], where visual comparisons were made between these wavelets. In this work, a more detailed comparative analysis between Golden and Mexican Hat wavelets is performed, highlighting the similarities and distinctions between them, both in relation to their waveforms and their applicability in time-frequency analysis. Therefore, in this work, it is intended to present the Golden Hat as an alternative to Mexican Hat, which, due to its characteristics, may be more advantageous in certain applications.

2 Real Wavelets and Continuous Wavelet Transform

A real wavelet $\psi(t)$ is a finite energy function, i.e., $\|\psi\|^2 = \int_{-\infty}^{+\infty} |\psi(t)|^2 dt < +\infty$, which satisfies the admissibility condition given by [1, 5]:

$$c_\psi = \int_0^{+\infty} \frac{|\Psi(j\Omega)|^2}{\Omega} \, d\Omega < +\infty, \tag{1}$$

where $\Psi(j\Omega) = \int_{-\infty}^{+\infty} \psi(t) \, e^{-j\Omega t} \, dt$ is the Fourier Transform (FT) of $\psi(t)$, where Ω is the angular frequency parameter, and $j = \sqrt{-1}$ is the imaginary unit. The function $\psi(t)$ is called a *mother wavelet* provided it is well localized and oscilatting [8]. The admissibility condition implies that $\int_{-\infty}^{+\infty} \psi(t) dt = \Psi(0) = 0$, which means that wavelets are zero average functions [2]. Furthermore, a wavelet can be interpreted as a bandpass linear filter [7].

For scaling s $(s > 0)$ and translation τ parameters, and for a specific real wavelet choice $\psi(t)$, the continuous wavelet transform (CWT) of a continuous time signal of finite energy $x(t)$ is given in the equation (2) [2, 7, 8]:

$$W(s, \tau) = \int_{-\infty}^{+\infty} x(t) \, \psi_{s,\tau}(t) \, dt, \tag{2}$$

where $\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}} \psi\left(\frac{t-\tau}{s}\right)$. As can see in equation (2), $W(s, \tau)$ measure, in certain sense, the fluctuations of the signal $x(t)$ around the point τ, at the scale given by $s > 0$ [8]. When s increases, $\psi(t)$ is expanded, and its frequency content moves to the lower frequency bands. Decreasing s implies the compression of $\psi(t)$, and its frequency content moves to the higher bands. Generally, wavelet functions are considered to have unit energy, i.e., $\|\psi\|^2 = 1$, and the normalization factor $1/\sqrt{s}$ ensures $\|\psi\|^2 = \|\psi_{s,\tau}\|^2$. Equation (2) can also be rewritten as a convolution product [7].

As can be seen from equation (2), $W(s, \tau)$ are not only affected by scale and translation parameters, but also by wavelet function choice. Therefore, the choice of the wavelet is very important and it depends on what signal features are desired to detect. There are different types of wavelet functions that can be used in the CWT. A typical wavelet choice is

$$\psi^{mh}(t) = \frac{2\,\pi^{-1/4}}{\sqrt{3}} \left(1 - t^2\right) e^{-t^2/2}, \tag{3}$$

that was first used in computer vision to detect multiscale edges [9]. This wavelet is the second derivative of the Gaussian probability density function, sometimes called the *Mexican Hat* function because it resembles a cross section of a "mexican hat" [2], as it can be seen in the Figure 1-(a) in red line, where $\psi^{mh}(t)$ plot is shown with normalized amplitude. The correspondent Mexican Hat FT is expressed by equation (4). The magnitude spectrum of the $\psi^{mh}(t)$ is shown in Figure 1-(b) in red line (also with normalized amplitude).

$$\Psi^{mh}(j\Omega) = \frac{\sqrt{8}\,\pi^{1/4}}{\sqrt{3}}\,\Omega^2 e^{-\Omega^2/2}. \tag{4}$$

3 Golden Hat: a new wavelet function

Recently, a new family of wavelet functions generated from FCPs was presented in [4]. This family is called Golden Wavelets, and each member is obtained by the n-th derivative of the quotient between two distinct FCPs. The Golden Hat that we will be defined and explored in this work belongs to that family, which contains other wavelets with different waveforms. See [4] for more details about the Golden family.

The FCPs were defined and introduced by [3], being constructed from the Fibonacci sequence. The first two terms of this sequence are $F_0 = 1$ and $F_1 = 1$, and the other terms are obtained recursively by $F_k = F_{k-1} + F_{k-2}$, $k \geq 2$. The polynomial sequence $\{p_n(t)\}_{n=0}^{\infty}$, defined by $p_n(t) = \sum_{k=0}^{n} F_{k+1} t^{n-k}$, is called Fibonacci-coefficient polynomial sequence, by setting $p_0(t) = 1$. We define the Golden Hat as the fourth derivative of the quotient between $p_0(t) = 1$ and $p_2(t) = t^2 + t + 2$, expressed by the equation:

$$g(t) = \frac{24(5t^4 + 10t^3 - 10t^2 - 15t - 1)}{(t^2 + t + 2)^5}. \tag{5}$$

In order to show that $g(t)$ is a wavelet, it must satisfy the admissibility condition defined by equation (1). To verify this, one must first calculate the FT of $g(t)$. For this purpose, it is extremely useful to use the FT time derivatives property [7]:

$$G(j\Omega) = (j \times \Omega)^4 H(j\Omega), \tag{6}$$

where $H(j\Omega)$ is the FT of the function $h(t)$ defined by the quotient between $p_0(t)$ and $p_2(t)$. Thus, $H(j\Omega)$ can be given by

$$H(j\Omega) = \int_{-\infty}^{+\infty} \frac{4}{(2t+1)^2 + 7} e^{-j\Omega t} dt. \tag{7}$$

Using integration techniques to solve (7):

$$H(j\Omega) = \frac{2\pi}{\sqrt{7}} e^{1/2\left(j\Omega - \sqrt{7}|\Omega|\right)}. \tag{8}$$

Therefore, by means of equations (8) and (6), we get

$$G(j\Omega) = \frac{2\pi}{\sqrt{7}} \Omega^4 e^{1/2\left(j\Omega - \sqrt{7}|\Omega|\right)}. \tag{9}$$

Finally, using $G(j\Omega)$ to calculate the admissibility condition defined in equation (1), we obtain: $c_g = \frac{2880\pi^2}{2401} < +\infty$, showing that $g(t)$ is a wavelet. In equation (10) we express the Golden Hat wavelet in a more simplified manner. In this expression, the Golden Hat is symmetric in $t = 0$ and has unit energy.

$$\psi^{gh}(t) = \frac{784 \times \sqrt{2} \times 7^{1/4} \times (80t^4 - 280t^2 + 49)}{\sqrt{5\pi} \times (4t^2 + 7)^5}. \tag{10}$$

In Figures 1-(a) and 1-(b) (blue lines) $\psi^{gh}(t)$ waveform and its magnitude spectrum $|\Psi^{gh}(j\Omega)|$ are shown, respectively. As well as Mexican Hat, Golden Hat resembles a cross section of a "mexican hat".

4 Comparative Analysis

Comparative analysis were performed imposing to Mexican and Golden Hat wavelets to have unit energy and an approximate support. In this way, one can better distinguish the differences between them. For this reason, it was fixed a unit scale for Mexican Hat, and it was chosen a specify scale $s_0 = 2.3265$ to dilate Golden Hat. In Figure 1, the wavelets $\psi^{mh}_{1,0}(t)$ and $\psi^{gh}_{s_0,0}(t)$ are plotted in the same plane, in their respective time (1-(a)) and frequency (1-(b)) domains.

As it can be seen in Figure 1-(a), the considered wavelets have a very similar waveform and they are symmetrical around $t = 0$. The similarity between this wavelets can also be seen through the global maximum and minimum values, and their respective localizations. In Table 1 these values and others metrics (with some approximate values) in time and frequency domains for both wavelets.

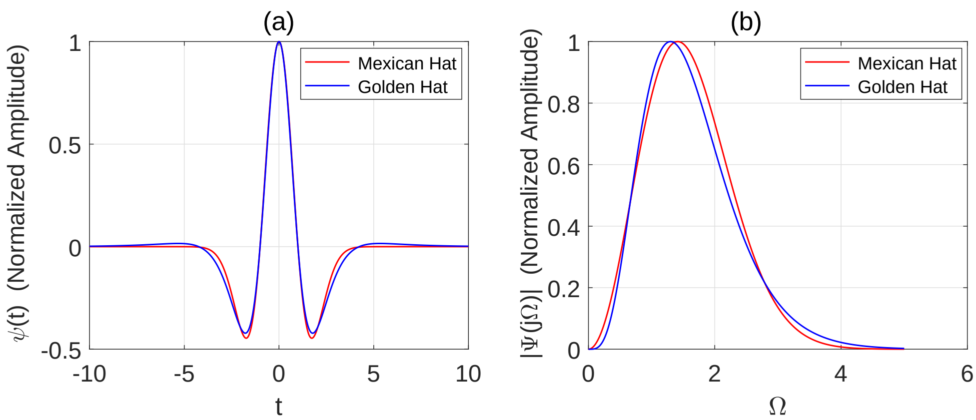

Figure 1: Wavelet waveforms in (a) time domain and (b) Magnitude spectrum in frequency domain. Mexican Hat and Golden Hat are plotted as red trace and blue trace, respectively, both with normalized amplitudes.

In relation to time domain, Golden Hat has four real zeros, while the Mexican Hat has only two zeros. This fact explains why Golden Hat has more oscillations, which implies

Table 1: Metrics for both wavelets.

Metrics	**Golden Hat**	**Mexican Hat**
Global Maximum	~ 0.86 at $t = 0$	~ 0.86 at $t = 0$
Global Minimum	~ -0.36 at $t \cong \pm 1.77$	~ -0.38 at $t \cong \pm 1.73$
Zeros	$t \cong \pm 1.00$ and $t \cong \pm 4.23$	$t = \pm 1.00$
p	4	2
μ	0.00	0.00
σ_t^2	~ 1.35	~ 1.17
η	~ 1.46	~ 1.50
σ_Ω^2	~ 0.24	~ 0.24
Peak frequency	$\Omega \cong 1.29$	$\Omega \cong 1.41$
Passband	$0.67 - 2.22$	$0.68 - 2.31$
Bandwidth	1.55	1.63

a greater number p of vanishing moments. A wavelet $\psi(t)$ has p vanishing moments if the following condition is verified [8]: $\int_{-\infty}^{+\infty} t^k \psi(t) dt = 0$, $k = 0, 1, \cdots, p-1$. It can also be seen that when $|t| \to +\infty$, the functions $\psi^{gh}(t) \to 0$ and $\psi^{mh}(t) \to 0$. Since this decay is exponential, both wavelets have an effective support that will depend on the specific tolerance in each application. For a wavelet with $\|\psi\|^2 = 1$, time localization μ is defined by $\mu = \int_{-\infty}^{+\infty} t|\psi(t)|^2 dt$, see [7]. Thus, using such expression, we find that the time localization for both wavelets is $\mu = 0$. The parameter σ_t^2 indicates the spread around μ, calculated by the variance $\sigma_t^2 = \int_{-\infty}^{+\infty} |t-\mu|^2 |\psi(t)|^2 dt$. Note that the values of σ_t^2 indicate that $\psi^{gh}(t)$ has more time dispersion, i.e., the wavelet $\psi^{gh}(t)$ is less localized in time domain than $\psi^{mh}(t)$.

In the frequency domain, the considered metrics were analized only to positive frequencies. This is done only to characterize the wavelets frequency response. The η and σ_Ω^2 parameters are the wavelet center frequency and the spread around it, respectively. These values are measured by $\eta = \frac{1}{2\pi} \int_0^{+\infty} \Omega |\Psi(j\Omega)|^2 d\Omega$ and $\sigma_\Omega^2 = \frac{1}{2\pi} \int_0^{+\infty} (\Omega - \eta)^2 |\Psi(j\Omega)|^2 d\Omega$ [7]. These parameters are relationed with analytic complex wavelets, since $\Psi(j\Omega) = 0$ if $\Omega < 0$, see [7]. Since the magnitude spectrum of the wavelets are asymmetric, the center frequency is not the peak frequency for both $\psi^{gh}(t)$ and $\psi^{mh}(t)$. It is noteworthy that σ_t^2 values indicate that wavelets have approximately the same frequency dispersion. The passband and bandwidth were characterize by the magnitude spectrum with normalized amplitude. The passband indicated in Table 1 was calculated using a cut-off frequency of 6 dB attenuation (close to 0.5 relative to peak). The filter bandwidth is simply the difference between the passband frequencies upper and lower limit. It is observed that Mexican Hat bandwidth is greater than the one for Golden Hat.

The comparative analysis was performed considering a Golden Hat dilatation by a factor of s_0 since, in this way, the wavelets waveforms are approximately superimposed. If this expansion were not performed, Golden Hat would present very different metrics in relation to Mexican Hat, since the Golden Hat generation gives it a greater compact support. Therefore, the results presented in Table 1 vary according to the scaling parameter

s considered.

In order to show a decomposition example for the considered wavelets, CWT was used to decompose a artificial signal $x(t)$. In Figure 2-(a), its shown the signal $x(t)$ with four singularities at $t = 0.2$, $t = 0.4$, $t = 0.6$ and $t = 0.8$. For this decomposition, the wavelets $\psi^{mh}(t)$ and $\psi^{gh}_{s_0,0}(t)$ were the chosen as wavelet mother. In Figure 2-(b) it is shown the decomposition at 7-th scale, for both wavelets. In Figures 2-(c) and 2-(d) present the Mexican Hat and Golden Hat scalograms, respectively. These scalograms are obtained by the CWT decomposition up to the 7-th scale. Numerical implementation was used to this CWT decomposition.

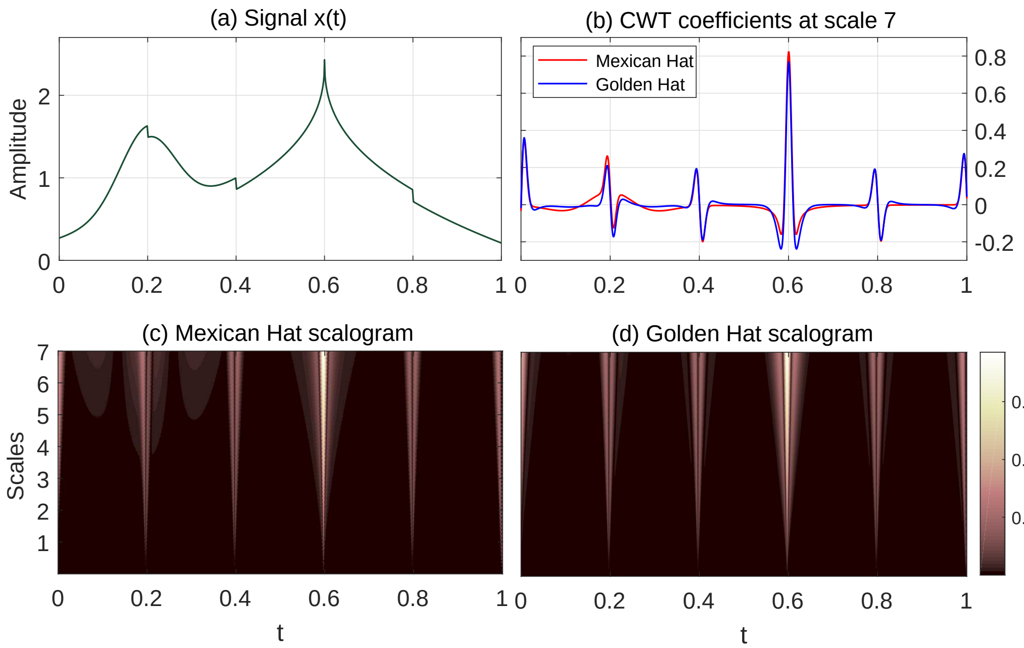

Figure 2: Graphics of (a) analized signal $x(t)$ and (b) WT coefficients at 7-th scale, calcuated by Mexican Hat (red line) and Golden Hat (blue line); escalograms for (c) Mexican Hat and (d) Golden Hat.

From Mexican and Golden Hat scalograms, it can be observed a wide similarity between them. Note that, both wavelets are efficient to detect and localize the singularities at low scales. This similarity is expected, since the wavelets have a very similar waveform. By the way, its observed small changes in the scalograms, as can been seen at high scales in the region of $t = 0.1$ and $t = 0.3$ (see Figure 2-(b)). For these scales, Mexican Hat was able to detect smooth signal features, since the Golden Hat not detected it. The differences for both decompositions may be related by the vanishing moments. More vanishing moments means that wavelet is smoother, and this is directly related to measure the local regularity of a signal, see [7]. Therefore, in relation to scales considered, the Golden Hat measured purely the regularity of the signal. On the other hand, if larger scales are considered, smooth signal features will be detected for both wavelets.

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5 Conclusions

In this work comparisons between Golden and Mexican Hat wavelets were performed showing a great similarity between them, according to the metrics considered in the comparative analysis, both in time and frequency domains. However, the Golden Hat wavelet have twice as much vanishing moments than Mexican Hat. Thus, since the local regularity of a signal is linked to the number of vanishing moments of a wavelet, the Golden Hat can be chosen as an alternative to Mexican Hat depending on the characteristics of the application, giving more softness to the wavelet signal representation.

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References

[1] A. P. Calderón. Intermediate spaces and interpolation: the complex method. *Studia Mathematica*, 24:113-190, 1964.

[2] I. Daubechies. *Ten lectures on wavelets*. Society for Industrial and Applied Mathematics, Philadelphia, 1992.

[3] D. Garth, D. Mills and P. Mitchell. Polynomials generated by the Fibonacci sequence, *Journal of Integer Sequences*, 10:1-12, 2007.

[4] F. E. Gossler, Wavelets e polinômios com coeficientes de Fibonacci, Dissertação de Mestrado, Unesp, 2016.

[5] A. Grossmann and J. Morlet. Decomposition of Hardy functions into square integrable wavelets of constant shape, *SIAM journal on mathematical analysis*, 15(4):723-736, 1984.

[6] R. L. Lamblém, F. E. Gossler, M. A. Q. Duarte and F. Villarreal, Polinômios com coeficientes de Fibonacci e wavelets, *Proceeding Series of the Brazilian Society of Applied and Computational Mathematics*, volume 1, 2017. DOI: 10.5540/03.2017.005.01.0036.

[7] S. Mallat. *A wavelet tour of signal processing: the sparse way.* Academic, Burlington, 2008.

[8] Y. Meyer. *Wavelets: algorithms and applications.* Society for Industrial and Applied Mathematics, Philadelphia, 1993.

[9] A. Witkin. *Scale space filtering.* Proc. Int. Joint. Conf. Artificial Intell., 1983.

