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Mathematical model of Diffusion-Advection for Zika

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Abstract. Zika virus spreads to people primarily through the bite of an infected Aedes species mosquito (Ae. aegypti and Ae. albopictus). Zika can also be passed through sex from a person who has Zika to his or her sex partners and it can be spread from a pregnant woman to her fetus. People can protect themselves from mosquito bites and getting Zika through sex. ZIKV continues to spread geographically to areas where competent vectors are present. Although a decline in cases of Zika virus infection has been reported in some countries, or in some parts of countries, vigilance needs to remain high. In this work we propose a mathematical model that uses diffusion-advection equations to study the impact of the Zika epidemic. We present a numerical scheme linked to the finite elements method (FEM) with finite differences to solve the model. The computer simulations are performed for Suriname and El Salvador which have different characteristics and allow us to extend the study to other regions.

Keywords. Difussion, epidemic, model, Zika.

1 Introduction

Zika virus is a mosquito-borne flavivirus that was first identified in Uganda in 1947 in monkeys through a network that monitored yellow fever. It was later identified in humans in 1952 in Uganda and the United Republic of Tanzania. Outbreaks of Zika virus disease have been recorded in Africa, the Americas, Asia and the Pacific. From the 1960s to 1980s, human infections were found across Africa and Asia, typically accompanied by mild illness. The first large outbreak of disease caused by Zika infection was reported from the Island of Yap (Federated States of Micronesia) in 2007 [5].
Zika virus is primarily transmitted to people through the bite of an infected mosquito from the Aedes genus, mainly Aedes aegypti in tropical regions. Aedes mosquitoes usually bite during the day, peaking during early morning and late afternoon/evening. This is the same mosquito that transmits dengue, chikungunya and yellow fever. Sexual transmission of Zika virus is also possible [4].
Recovery from Zika virus disease may require anywhere from 3 to 14 days after becoming infectious, but once recovered humans are believed to be immune from the virus for life

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many people infected with Zika may be asymptomatic or will only display mild symptoms that do not require medical attention [4].

The use of diffusion and advection-diffusion equations in the study of epidemics can be seen in [10,11], in particular for Dengue [8,9,14], for HIV / AIDS in [6,12] and for Malaria in [7], these texts contributed background in the work that we present.

The objective of this work is to present a model for the Zika epidemic based on the Diffusion-advection equations. To solve this model we use a numerical scheme based on the finite elements method (FEM) and finite differences. Computational simulations are performed in Surinam and El Salvador.

2 Mathematical Model

In the model that we propose, the variables are susceptible humans (S), infected humans (I), recovered humans (R), susceptible mosquitoes (M) and infected mosquitoes (P).
Assumptions:

- We accept immunity: from the recovered state does not possible to return to the susceptible state.
- There is not consideration of vertical transmission in humans or mosquitoes.
- The death by natural causes is equal in any state.
- The death of mosquitoes will be due to environmental factors because no control strategy is applied.
- By definition epidemiological S, I, R, M and P are positive or null.
- The study was restricted to null border conditions.

The formulation of the model is:

$$\frac{\partial S}{\partial t} - \alpha_s \Delta S + \beta_s \nabla S = N_1 - \beta_{y_1} SP - \mu S, \tag{1}$$

$$\frac{\partial I}{\partial t} - \alpha_I \Delta I + \beta_I \nabla I = \beta_{y_1} SP - (r + \epsilon + \mu) I, \tag{2}$$

$$\frac{\partial R}{\partial t} - \alpha_r \Delta R + \beta_r \nabla R = rI - \mu R, \tag{3}$$

$$\frac{\partial M}{\partial t} - \alpha_l \Delta M + \beta_l \nabla M = N_4 - \beta_x MI - \xi M, \tag{4}$$

$$\frac{\partial P}{\partial t} - \alpha_p \Delta P + \beta_p \nabla P = \beta_x MI - \xi P. \tag{5}$$

Initial Conditions

$$t \in [0, t_f],$$

$$S(0) = s_0 > 0, \quad I(0) = i_0 > 0, \quad R(0) = r_0 \geq 0, \quad M(0) = l_0 \geq 0 \quad P(0) = p_0 \geq 0,$$
$$\frac{\partial S}{\partial \eta} = 0, \quad \frac{\partial I}{\partial \eta} = 0, \quad \frac{\partial R}{\partial \eta} = 0, \quad \frac{\partial M}{\partial \eta} = 0 \quad \frac{\partial P}{\partial \eta} = 0.$$

The elements of the model are describen on the below table (1).

Table 1: Description of parameters used in the model.

Parameters	Description
α_s	Dispersion rate of susceptible humans
α_I	Dispersion rate of infected humans
α_r	Dispersion rate of recovered humans
α_l	Dispersion rate of susceptible mosquitoes
α_p	Dispersion rate of indected mosquitoes
β_s	Advective transport rate of susceptible humans
β_I	Advective transport rate of infected humans
β_r	Advective transport rate of recovered humans
β_l	Advective transport rate of susceptible mosquitoes
β_p	Advective transport rate of infected mosquitoes
β_x	The force of infection from infected human to susceptible mosquito
β_{y_1}	The force of infection from infected mosquito to susceptible human
μ , η	Human and mosquito natural death rates
ϵ	Disease-induced death rate for humans
r	Human recovery rate
N_1	Entry rate of susceptible humans
N_4	Entry rate of susceptible mosquitoes

2.1 Method of solution

First we find the variational formulation of the model and apply the Garlekin method, [2], [3].
Let $W = \{L^2([0, T_f], V)\}$, $V = H^1(W)$ space of test functions and in V we define the scalar product:

$$\langle u, v \rangle = \int_\Omega uvdxdy, \qquad \langle \nabla u || \nabla v \rangle = \int_\Omega \nabla u \nabla v d\mu$$

$u \in W$, $v \in V$.
Let $U = S, I, R, M, P$, $U(x, y, t) = U$, $v = v(x, y)$, by the formula of Green and for border conditions, $\dfrac{\partial U}{\partial \eta} = 0$ in $\Omega = [0, t_f]$, we have

$$-\alpha_U \langle \Delta U, v \rangle = \alpha_U \langle \nabla U || \nabla v \rangle.$$

Let $\{\phi_i\}$ base of V and we are going to write:

$$\beta_s = \langle \beta_{s_1}, \beta_{s_2} \rangle, \quad \beta_I = \langle \beta_{I_1}, \beta_{I_2} \rangle, \quad \beta_r = \langle \beta_{r_1}, \beta_{r_2} \rangle, \quad \beta_l = \langle \beta_{l_1}, \beta_{l_2} \rangle, \quad \beta_p = \langle \beta_{p_1}, \beta_{p_2} \rangle.$$

In this approximation the model is expressed as:

$$
\begin{aligned}
&\sum_j \frac{dS_j}{dt}\langle\phi_j,\phi_i\rangle - \alpha_s \sum_j S_j \langle\nabla\phi_j||\nabla\phi_i\rangle + \beta_{s_1}\sum_j S_j\langle\frac{\partial\phi_j}{\partial x},\phi_i\rangle + \beta_{s_2}\sum_j S_j\langle\frac{\partial\phi_j}{\partial y},\phi_i\rangle\\
&= N_1 - \beta_{y_1}\sum_j P_j \sum_k S_k\langle\phi_j\phi_k,\phi_i\rangle - \mu\sum_j S_j\langle\phi_j,\phi_i\rangle,\\
&\sum_j \frac{dI_j}{dt}\langle\phi_j,\phi_i\rangle - \alpha_I \sum_j I_j \langle\nabla\phi_j||\nabla\phi_i\rangle + \beta_{I_1}\sum_j I_j\langle\frac{\partial\phi_j}{\partial x},\phi_i\rangle + \beta_{I_2}\sum_j I_j\langle\frac{\partial\phi_j}{\partial y},\phi_i\rangle\\
&= \beta_{y_1}\sum_j P_j \sum_k S_k\langle\phi_j\phi_k,\phi_i\rangle - (r+\mu+\epsilon)\sum_j I_j\langle\phi_j,v\rangle,\\
&\sum_j \frac{dR_j}{dt}\langle\phi_j,\phi_i\rangle - \alpha_r \sum_j R_j \langle\nabla\phi_j||\nabla\phi_i\rangle + \beta_{r_1}\sum_j R_j\langle\frac{\partial\phi_j}{\partial x},\phi_i\rangle + \beta_{r_2}\sum R_j\langle\frac{\partial\phi_j}{\partial y},\phi_i\rangle\\
&= r\sum_j I_j\langle\phi_j,\phi_i\rangle - \mu\sum_j R_j\langle\phi_j,v\rangle,\\
&\sum_j M_j\langle\phi_j,v\rangle - \alpha_l \sum_j M_j \langle\nabla\phi_j||\nabla\phi_i\rangle + \beta_{l_1}\sum_j M_j\langle\frac{\partial\phi_j}{\partial x},v\rangle + \beta_{l_2}\sum_j M_j\langle\frac{\partial\phi_j}{\partial y},\phi_i\rangle\\
&= N_4 - \beta_x\sum_j I_j \sum_k M_k\langle\phi_j\phi_k,\phi_i\rangle - \xi\sum_j M_j\langle\phi_j,\phi_i\rangle,\\
&\sum_j P_j\langle\phi_j,\phi_i\rangle - \alpha_p \sum_j P_j \langle\nabla\phi_j||\nabla\phi_i\rangle + \beta_{p_1}\sum_j P_j\langle\frac{\partial\phi_j}{\partial x},\phi_i\rangle + \beta_{p_2}\sum_j p_j\langle\frac{\partial\phi_j}{\partial y},\phi_i\rangle\\
&= \beta_x\sum_j I_j \sum_k M_k\langle\phi_j\phi_k,\phi_i\rangle - \xi\sum_j p_j\langle\phi_j,\phi_i\rangle.
\end{aligned}
$$

For the temporary variables the Crank-Nicolson method (central differences in the time $t_{\frac{n+1}{2}}$) was used and the scheme is, [1],

$$
\begin{aligned}
&\sum_j\left(\frac{S_j^{n+1}-S_j^n}{\Delta t}\right)\langle\phi_j,\phi_i\rangle - \alpha_s\sum_j\left(\frac{S_j^{n+1}+S_j^n}{2}\right)\langle\nabla\phi_j||\nabla\phi_i\rangle + \beta_{s_1}\sum_j\left(\frac{S_j^{n+1}+S_j^n}{2}\right)\langle\frac{\partial\phi_j}{\partial x},\phi_i\rangle\\
&+\beta_{s_2}\sum_j\left(\frac{S_j^{n+1}+S_j^n}{2}\right)\langle\frac{\partial\phi_j}{\partial y},\phi_i\rangle\beta_{s_2}\sum_j\left(\frac{S_j^{n+1}+S_j^n}{2}\right)\langle\frac{\partial\phi_j}{\partial y},\phi_i\rangle - \mu\sum_j\left(\frac{S_j^{n+1}+S_j^n}{2}\right)\langle\phi_j,\phi_i\rangle.\\
&\sum_j\left(\frac{I_j^{n+1}-I_j^n}{\Delta t}\right)\langle\phi_j,\phi_i\rangle - \alpha_I\sum_j\left(\frac{I_j^{n+1}+I_j^n}{2}\right)\langle\nabla\phi_j||\nabla\phi_i\rangle + \beta_{I_1}\sum_j\left(\frac{I_j^{n+1}+I_j^n}{2}\right)\langle\frac{\partial\phi_j}{\partial x},\phi_i\rangle\\
&+\beta_{I_2}\sum_j\left(\frac{I_j^{n+1}+I_j^n}{2}\right)\langle\frac{\partial\phi_j}{\partial y},\phi_i\rangle = \beta_{y_1}\sum_j\left(\frac{P_j^{n+1}+P_j^n}{2}\right)\sum_k\left(\frac{S_k^{n+1}+S_k^n}{2}\right)\langle\phi_j\phi_k,\phi_i\rangle\\
&-(r+\mu+\epsilon)\sum_j\left(\frac{I_j^{n+1}+I_j^n}{2}\right)\langle\phi_j,\phi_i\rangle.\\
&\sum_j\left(\frac{R_j^{n+1}-R_j^n}{\Delta t}\right)\langle\phi_j,\phi_i\rangle - \alpha_r\sum_j\left(\frac{R_j^{n+1}+R_j^n}{2}\right)\langle\nabla\phi_j||\nabla\phi_i\rangle + \beta_{r_1}\sum_j\left(\frac{R_j^{n+1}+R_j^n}{2}\right)\langle\frac{\partial\phi_j}{\partial x},\phi_i\rangle
\end{aligned}
$$

$$+\beta_{r_2}\sum_j\left(\frac{R_j^{n+1}+R_j^n}{2}\right)\langle\frac{\partial\phi_j}{\partial y},\phi_i\rangle = r\sum_j\left(\frac{I_j^{n+1}+I_j^n}{2}\right)\langle\phi_j,\phi_i\rangle-\mu\sum_j\left(\frac{R_j^{n+1}+R_j^n}{2}\right)\langle\phi_j,\phi_i\rangle.$$

$$\sum_j\left(\frac{M_j^{n+1}-M_j^n}{\Delta t}\right)\langle\phi_j,\phi_i\rangle-\alpha_l\sum_j\left(\frac{M_j^{n+1}+M_j^n}{2}\right)\langle\nabla\phi_j||\nabla\phi_i\rangle+\beta_{l_1}\sum_j\left(\frac{M_j^{n+1}+M_j^n}{2}\right)\langle\frac{\partial\phi_j}{\partial x},\phi_i\rangle$$

$$+\beta_{l_2}\sum_j\left(\frac{M_j^{n+1}+M_j^n}{2}\right)\langle\frac{\partial\phi_j}{\partial y},\phi_i\rangle = N_4-\beta_x\sum_j\left(\frac{I_j^{n+1}+I_j^n}{2}\right)\sum_k\left(\frac{M_k^{n+1}+M_k^n}{2}\right)\langle\phi_j\phi_k,\phi_i\rangle$$

$$-\eta\sum_j\left(\frac{M_j^{n+1}+M_j^n}{2}\right)\langle\phi_j,\phi_i\rangle.$$

$$\sum_j\left(\frac{P_j^{n+1}-P_j^n}{\Delta t}\right)\langle\phi_j,\phi_i\rangle-\alpha_p\sum_j\left(\frac{P_j^{n+1}+P_j^n}{2}\right)\langle\nabla\phi_j||\nabla\phi_i\rangle+\beta_{p_1}\sum_j\left(\frac{P_j^{n+1}+P_j^n}{2}\right)\langle\frac{\partial\phi_j}{\partial x},\phi_i\rangle$$

$$+\beta_{p_2}\sum_j\left(\frac{P_j^{n+1}+P_j^n}{2}\right)\langle\frac{\partial\phi_j}{\partial y},\phi_i\rangle = \beta_x\sum_j\left(\frac{I_j^{n+1}+I_j^n}{2}\right)\sum_k\left(\frac{M_k^{n+1}+M_k^n}{2}\right)\langle\phi_j\phi_k,\phi_i\rangle$$

$$-\eta\sum_j\left(\frac{P_j^{n+1}+P_j^n}{2}\right)\langle\phi_j,\phi_i\rangle.$$

The matrix formulation for the iterative process is:

$$\begin{aligned}
&C_s(P^{n+1},P^n)S^{n+1} = D_s(P^{n+1},P^n)S^n,\\
&C_I(c)I^{n+1} = D_I(S^n,S^{n+1},P^n,P^{n+1})I^n,\\
&C_R(c)R^{n+1} = DR(I^n,I^{n+1})R^n,\\
&C_M(I^n,I^{n+1})M^{n+1} = D_M(I^n,I^{n+1})M^n,\\
&C_P(c)P^{n+1} = D_P(I^n,I^{n+1},M^n,M^{n+1})P^n.
\end{aligned}$$

Where $C_I(c)$, $C_R(c)$ and $C_P(c)$ are matrices of constant coefficients.
Solution algorithm proposal:
The sistem for $n=0$,

$$\begin{aligned}
&C_s(P^{n+1},P^n)S^1 = D_s(P^{n+1},P^n)S^0,\\
&C_I(c)I^1 = D_I(S^n,S^{n+1},P^n,P^{n+1})I^0,\\
&C_R(c)R^1 = D_R(I^n,I^{n+1})R^0,\\
&C_M(I^n,I^{n+1})M^1 = D_M(I^n,I^{n+1})M^0,\\
&C_P(c)P^1 = D_P(I^n,I^{n+1},M^n,M^{n+1})P^0.
\end{aligned}$$

To solve the system we used a predictor-corrector method, [**?**], [**?**].
We start with the initial conditions (S^0,I^0,R^0,M^0,P^0), and we look for the first approximation (S^1,I^1,R^1,M^1,P^1).
For S^* we solve: $C_s(P^0,P^0)S^* = D_s(P^0,P^0)S^0$.
Then look for I^*, $C_I(c)I^* = D_I(S^0,S^*,P^0,P^0)I^0$.
We calculate R^*, $C_R(c)R^* = D_R(I^0,I^*)R^0$.
We calculate M^*, $C_M(I^0,I^*)M^* = D_M(I^0,I^*)M^0$.
We calculate P^*, $C_P(c)P^* = D_P(I^0,I^*,M^0,M^*)P^0$.
We obtained (S^*,I^*,R^*,M^*,P^*).

Now, with $(S^0, I^0, R^0, M^0, P^0)$ and $(S^*, I^*, R^*, M^*, P^*)$ we look for $(S^{**}, I^{**}, R^{**}, M^{**}, P^{**})$, analogously to the previous scheme.
We calculate S^{**} solving: $C_s(P^0, P^*)S^{**} = D_s(P^0, P^*)S^0$.
We calculate I^{**}, $C_I(c)I^{**} = D_I(S^0, S^{**}, P^0, P^*)I^0$.
We calculate R^{**}, $C_R(c)R^{**} = D_R(I^0, I^{**})R^0$.
We calculate M^{**}, $C_M(I^0, I^*)M^{**} = D_M(I^0, I^*)M^0$.
We calculate P^{**}, $C_P(c)P^{**} = D_P(I^0, I^{**}, M^0, M^{**})P^0$.
We declare the conditions of convergence and we continue the process. Then, for $p\ in \mathbb{N}$:

$$(S^1 = S^{n*}, I^1 = I^{n*}, R^1 = R^{n*}, M^1 = M^{n*}, P^1 = P^{n*}).$$

Repeat the process for n y $(S^n, I^n, R^n, M^n, P^n)$ is the approximate solution of the model.

3 Discussion

The simulations are carried out for Suriname and El Salvador, which have different characteristics (in Suriname $\beta_{y_1} > \beta_x$ and in El Salvador $\beta_x > \beta_{y_1}$) and Zika can become an endemic problem. The values of parameters were extracted from [**?**, 8, 13, 14] and Matlab-R2017a software was used for programming. The unit of time is months and the initial conditions were taken from [13]. All subpopulations were studied. Between the results of the experimentation is, El Salvador reports increased diffusion of infected people respect to Suriname which shows that the β_x parameter has a strong influence on the model.

4 Conclusions

We present a mathematical model for the Zika epidemic that allows us to study its diffusion over time. A numerical scheme was developed with the use of FEM and Crank-Nicolson to solve the model. Computational experimentation showed the impact of the epidemic in Suriname and El Salvador, where Zika can become endemic.

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