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# An Introduction to the Complete Lattice Projection Autoassociative Memories: Definitions and Examples

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**Abstract**. An autoassociative memory is an input-output system designed for the storage and recall of a finite set of items. In this work, we present the class of complete lattice projection autoassociative memories (CLPAMs). A CLPAM is a non-distributive autoassociative memory defined by a neural network with a hidden layer of morphological neurons. More importantly, a CLPAM is formulated using only the partial ordering of a complete lattice. As an example of CLPAM, we introduce the so called distance-based projection autoassociative memories (DBPAMs) which exhibit an excellent tolerance to salt-and-pepper noise.

**Keywords**. Associative memory, morphological neural networks, lattice computing, pattern reconstruction.

## 1 Introduction

Associative memories are mathematical structures inspired by the human brain ability to store and recall information. From a mathematical point of view, an associative memory is a mapping $\mathcal{M}$ designed for the storage of a finite set of association pairs $\{(\mathbf{x}^1, \mathbf{y}^1), \ldots, (\mathbf{x}^k, \mathbf{y}^k)\}$ [5]. Furthermore, an associative memory $\mathcal{M}$ is expected to retrieve a memorized item $\mathbf{y}^\xi$ even upon the presentation of a corrupted or partial version $\tilde{\mathbf{x}}^\xi$ of key item $\mathbf{x}^\xi$. Equivalently, the identity $\mathcal{M}(\tilde{\mathbf{x}}^\xi) = \mathbf{y}^\xi$ is expected to hold true for a partial or corrupted version $\tilde{\mathbf{x}}^\xi$ of $\mathbf{x}^\xi$, for all $\xi \in \{1, \ldots, k\}$. On the one hand, we say that $\mathcal{M}$ is a heteroassociative memory if $\mathbf{x}^\xi$ differs from $\mathbf{y}^\xi$ for at least one index $\xi \in \{1, \ldots, k\}$. On the other hand, we have an autoassociative memory $\mathcal{M}$ if $\mathbf{x}^\xi$ coincides with $\mathbf{y}^\xi$ for all $\xi = 1, \ldots, k$. In this paper we only consider autoassociative memory models designed for the storage and recall of a set finite set $\mathcal{X} = \{\mathbf{x}^1, \ldots, \mathbf{x}^k\}$.

The famous recurrent neural network of Hopfield is an example of a model that can be used to implement an autoassociative memory [6]. Apart from traditional models like

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the Hopfield's network, associative memory models based on lattice computing paradigm have been investigated since the middle 1990s [8]. According to Kaburlasos et al. [7], lattice computing is defined as an evolving collection of tools and mathematical modeling methodologies with the capacity to process lattice-ordered data *per se*, including logic values, numbers, sets, symbols, graphs, etc. Lattice computing-based associative memories include, for instance, the traditional gray-scale morphological associative memories [9], the broad class of fuzzy morphological associative memories [14, 15], and the Θ-fuzzy associative memories [3]. Recently, we introduced the class of the max-plus and min-plus projection autoassociative morphological memories (PAMMs), which also belongs to the lattice computing paradigm [12]. Apart from the low computational effort, lattice computing-based associative memories have been effectively applied for pattern classification [3, 12], times-series prediction [15], and image understanding and reconstruction [9, 4].

In the last year, we introduced the class of max-$C$ and min-$D$ projection autoassociative fuzzy memories (max-$C$ and min-$D$ PAFMs) [11, 13]. Briefly, these models grew out of a combination of the max-plus and min-plus projection autoassociative morphological memories and the fuzzy morphological associative memories. Precisely, a PFAM projects the input vector into either the set of all max-$C$ combinations or the set of all min-$D$ combinations of the stored vectors. In addition, using Gaines' fuzzy conjunction and fuzzy implication, we obtained the so-called max-$C$ PAFM of Zadeh [10]. The max-$C$ PAFM of Zadeh, as well as its dual model, does not perform any arithmetic operation. In other words, the max-$C$ and min-$D$ PAFM of Zadeh depend only on the partial ordering of the complete lattice $[0, 1]^n$.

Motivated by the remarks in the previous paragraph, in this paper we propose the class of complete lattice projection autoassociative memories (CLPAMs). CLPAMs are only based on a complete lattice structure and, thus, they belong to the lattice computing paradigm. Apart from defining the broad class of CLPAMs, we present the so-called distance-based projection autoassociative memories (DBPAMs). Besides the max-$C$ and min-$D$ PAFMs de Zadeh, the class of DBPAMs include models that exhibit an excellent tolerance to salt-and-pepper noise.

## 2 Complete Lattice Projection Autoassociative Memories

First of all, a complete lattice $\mathbb{L}$ is a nonempty set equipped with a partial order $\preccurlyeq_{\mathbb{L}}$ such that every subset $X \subset \mathbb{L}$ has a supremum and an infimum on $\mathbb{L}$ [2]. The symbols $\bigwedge X$ and $\bigvee X$ denote respectively the infimum and supremum of the set $X \subseteq \mathbb{L}$. In particular, the least and the largest elements of $\mathbb{L}$ are respectively $\bigwedge \mathbb{L} = \boldsymbol{O}_{\mathbb{L}}$ and $\bigvee \mathbb{L} = \boldsymbol{1}_{\mathbb{L}}$. Moreover, the supremum and the infimum of the empty set are $\bigvee \emptyset = \boldsymbol{O}_{\mathbb{L}}$ and $\bigwedge \emptyset = \boldsymbol{1}_{\mathbb{L}}$, respectively. Examples of complete lattice include extended real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ and extended integer numbers $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty, -\infty\}$ with the usual order, denoted in this paper by the symbol $\leq$.

Let $\mathcal{L}_1, \ldots, \mathcal{L}_n$ be complete lattices with partial orders $\preccurlyeq_{\mathcal{L}_1}, \ldots, \preccurlyeq_{\mathcal{L}_n}$, respectively. The Cartesian product $\mathbb{L} = \mathcal{L}_1 \times \ldots \times \mathcal{L}_n$ is also a complete lattice with the following induced ordering: Given $\mathbf{x} = [x_1, \ldots, x_n]^T$ and $\mathbf{y} = [y_1, \ldots, y_n]^T \in \mathbb{L}$, we define

$$\mathbf{x} \preccurlyeq_{\mathbb{L}} \mathbf{y} \quad \Longleftrightarrow \quad x_i \preccurlyeq_{\mathcal{L}_i} y_i, \quad \forall i = 1, \ldots, n. \tag{1}$$

Consider a set $\mathcal{X} = \{\mathbf{x}^1, \ldots, \mathbf{x}^k\}$, called fundamental memory set. An autoassociative memory is a mapping $\mathcal{M}$ such that the identity $\mathcal{M}(\mathbf{x}^\xi) = \mathbf{x}^\xi$ holds true as far as possible for all $\xi \in \mathcal{K} = \{1, 2, \ldots, k\}$. Moreover, an associative memory must exhibit some noise tolerance, that is, we expect $\mathcal{M}(\widetilde{\mathbf{x}}^\xi) = \mathbf{x}^\xi$ for a corrupted or partial version $\widetilde{\mathbf{x}}^\xi$ of the fundamental memory $\mathbf{x}^\xi$ [5].

Let $\mathbb{L}$ be a complete lattice and consider a set $\mathcal{X} = \{\mathbf{x}^1, \ldots, \mathbf{x}^k\} \subset \mathbb{L}$ of fundamental memories. A complete lattice projection autoassociative memory (CLPAM), denoted by $\mathcal{S}_{\mathbb{L}} : \mathbb{L} \longrightarrow \mathbb{L}$, is the associative memory defined as follows for any input $\mathbf{x} \in \mathbb{L}$:

$$\mathcal{S}_{\mathbb{L}}(\mathbf{x}) = \bigwedge_{\xi \in \mathcal{J}_{\mathbb{L}}} \mathbf{x}^\xi, \quad \text{where} \quad \mathcal{J}_{\mathbb{L}} = \left\{\xi \in \mathcal{K} : \mathbf{x} \preccurlyeq_{\mathbb{L}} \mathbf{x}^\xi\right\}. \tag{2}$$

Alternatively, the memory $\mathcal{S}_{\mathbb{L}}$ can be expressed by

$$\mathcal{S}_{\mathbb{L}}(\mathbf{x}) = \bigwedge \left\{\mathbf{x}^\xi \in \mathcal{X} : \mathbf{x} \preccurlyeq_{\mathbb{L}} \mathbf{x}^\xi\right\}. \tag{3}$$

In words, a CLPAM $\mathcal{S}_{\mathbb{L}}(\mathbf{x})$ is the greatest element of $\mathbb{L}$ which is less than or equal to all fundamental memories greater than the input $\mathbf{x}$. As a consequence, a CLPAM $\mathcal{S}_{\mathbb{L}}$ satisfies $\mathcal{S}_{\mathbb{L}}(\mathbf{x}^\xi) = \mathbf{x}^\xi$ for all $\xi \in \mathcal{K}$, i.e., the memory $\mathcal{S}_{\mathbb{L}}$ exhibits optimal absolute storage capacity. Furthermore, a CLPAM is an idempotent operator, i.e., $\mathcal{S}_{\mathbb{L}}(\mathcal{S}_{\mathbb{L}}(\mathbf{x})) = \mathcal{S}_{\mathbb{L}}(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{L}$. In other words, $\mathcal{S}_{\mathbb{L}}$ projects the input pattern $\mathbf{x}$ onto the minimum of the the fundamental memories such that $\mathbf{x} \preccurlyeq_{\mathbb{L}} \mathbf{x}^\xi$. Also, the associative memory $\mathcal{S}_{\mathbb{L}}$ is an extensive operator, that is, the inequality $\mathbf{x} \preccurlyeq_{\mathbb{L}} \mathcal{S}_{\mathbb{L}}(\mathbf{x})$ holds true for all $\mathbf{x} \in \mathbb{L}$. Summarizing, we have the following theorem:

**Theorem 2.1.** *Consider a fundamental memory $\mathcal{X} = \{\mathbf{x}^1, \ldots, \mathbf{x}^k\} \subset \mathbb{L}$, where $\mathbb{L}$ is a complete lattice. The associative memory $\mathcal{S}_{\mathbb{L}}$ defined by* (3) *is extensive ($\mathbf{x} \preccurlyeq_{\mathbb{L}} \mathcal{S}_{\mathbb{L}}(\mathbf{x})$), idempotent ($\mathcal{S}_{\mathbb{L}}(\mathcal{S}_{\mathbb{L}}(\mathbf{x})) = \mathcal{S}_{\mathbb{L}}(\mathbf{x})$), and satisfies $\mathcal{S}_{\mathbb{L}}(\mathbf{x}^\xi) = \mathbf{x}^\xi$ for all $\xi = 1, \ldots, k$.*

The following theorem address the noise tolerance of a CLPAM.

**Theorem 2.2.** *Consider a fundamental memory set $\mathcal{X} = \{\mathbf{x}^1, \ldots, \mathbf{x}^k\} \subset \mathbb{L}$ and let $\mathbf{x} \in \mathbb{L}$ be the input. If there exists only one index $\gamma \in \mathcal{K}$ such that $\mathbf{x} \preccurlyeq_{\mathbb{L}} \mathbf{x}^\gamma$, then the CLPAM defined by* (3) *satisfies $\mathcal{S}_{\mathbb{L}}(\mathbf{x}) = \mathbf{x}^\gamma$.*

Note that we cannot retrieve a fundamental memory $\mathbf{x}^\xi$ if $\mathbf{x}$ is greater than $\mathbf{x}^\xi$. We say that $\mathbf{x}$ is a dilated version of $\mathbf{x}^\xi$ if $\mathbf{x}^\xi \preccurlyeq_{\mathbb{L}} \mathbf{x}$. Dually, $\mathbf{x}$ is an eroded version of $\mathbf{x}^\xi$ if $\mathbf{x} \preccurlyeq_{\mathbb{L}} \mathbf{x}^\xi$. From Theorems 2.1 and 2.2, we conclude that a CLPAM $\mathcal{S}_{\mathbb{L}}$ is robust in the presence of erosive noise but it is not effective in the presence of dilative noise.

**Example 1.** Consider the set $\mathcal{L} = \{a, b, c, d\}$ equipped with the partial order depicted on Figure 1. Note that the inequalities $a \prec b \prec d$ and $a \prec c \prec d$ hold true. Although $b$ and $c$ are incomparable, we have $b \curlyvee c = d$ and $b \curlywedge c = a$. Thus, $\mathcal{L}$ constitutes a complete

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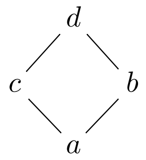


Figure 1: Hasse diagram of the complete lattice $\mathcal{L} = \{a, b, c, d\}$ of the Example 1.

lattice and, using the induced order, we have that $\mathbb{L} = \mathcal{L}^4$ is also a complete lattice. Now, consider the following set of fundamental memories:

$$\mathcal{X} = \left\{ \mathbf{x}^1 = \begin{bmatrix} d \\ b \\ c \\ c \end{bmatrix}, \mathbf{x}^2 = \begin{bmatrix} d \\ c \\ a \\ b \end{bmatrix}, \mathbf{x}^3 = \begin{bmatrix} b \\ a \\ c \\ b \end{bmatrix} \right\} \subset \mathbb{L}. \tag{4}$$

If $\mathbf{x} = \begin{bmatrix} c & b & c & a \end{bmatrix}^T$ is presented as input, the CLPAM $\mathcal{S}_{\mathbb{L}}$ defined by (3) yields

$$\mathcal{S}_{\mathbb{L}}(\mathbf{x}) = \curlywedge \left\{ \mathbf{x}^\xi \in \mathcal{X} : \mathbf{x} \preccurlyeq_{\mathbb{L}} \mathbf{x}^\xi \right\} = \curlywedge \left\{ \mathbf{x}^1 \right\} = \mathbf{x}^1, \tag{5}$$

because $\mathbf{x} \preccurlyeq_{\mathbb{L}} \mathbf{x}^1$ but $\mathbf{x} \not\preccurlyeq_{\mathbb{L}} \mathbf{x}^2$ and $\mathbf{x} \not\preccurlyeq_{\mathbb{L}} \mathbf{x}^3$.

In fuzzy systems, a fuzzy set on a finite universe of discourse can be identified with an element of the complete lattice $\mathbb{L} = [0, 1]^n$ with the usual partial order. In this case, the CLPAM $\mathcal{S}_{\mathbb{L}}$ coincides with the dual of the projection fuzzy autoassociative memory (PFAM) of Zadeh introduced recently by Santos and Valle [10].

**Example 2.** Let the hypercube $\mathbb{L} = [0, 1]^4$ be equipped with the usual order $\leq$. Consider the fundamental memory set

$$\mathcal{X} = \left\{ \mathbf{x}^1 = \begin{bmatrix} 0.6 \\ 0.7 \\ 0.1 \\ 0.7 \end{bmatrix}, \mathbf{x}^2 = \begin{bmatrix} 0.9 \\ 0.3 \\ 0.5 \\ 0.9 \end{bmatrix}, \mathbf{x}^3 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.6 \\ 0.8 \end{bmatrix} \right\}. \tag{6}$$

and the following input vectors

$$\mathbf{x} = \begin{bmatrix} 0.6 & 1.0 & 1.0 & 0.7 \end{bmatrix}^T \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 0.0 & 0.7 & 0.0 & 0.7 \end{bmatrix}^T, \tag{7}$$

which can be seen as versions of $\mathbf{x}^1$ corrupted by dilative and erosive noise, respectively. Upon the presentation of the vectors $\mathbf{x}$ and $\mathbf{y}$ as inputs, the CLPAM $\mathcal{S}_{\mathbb{L}}$ yields

$$\mathcal{S}_{\mathbb{L}}(\mathbf{x}) = \curlywedge \left\{ \mathbf{x}^\xi \in \mathcal{X} : \mathbf{x} \preccurlyeq_{\mathbb{L}} \mathbf{x}^\xi \right\} = \curlywedge \emptyset = [1.0, 1.0, 1.0, 1.0]^T, \tag{8}$$

and

$$\mathcal{S}_{\mathbb{L}}(\mathbf{y}) = \curlywedge \left\{ \mathbf{x}^\xi \in \mathcal{X} : \mathbf{x} \preccurlyeq_{\mathbb{L}} \mathbf{x}^\xi \right\} = \curlywedge \left\{ \mathbf{x}^1 \right\} = \mathbf{x}^1. \tag{9}$$

On the one hand, the CLPAM $\mathcal{S}_{\mathbb{L}}$ failed to retrieve the fundamental memory $\mathbf{x}^1$ upon the presentation of the vector $\mathbf{x}$ contaminated by dilative noise. On the other hand, the memory succeed to recall $\mathbf{x}^1$ from the version $\mathbf{y}$ of $\mathbf{x}^1$ corrupted by erosive noise.

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# 3 Distance-Based Projection Autoassociative Memories

Usually, we consider the unit interval $[0,1]$ with the usual order of real numbers. It turns out, however, that we can endow $[0,1]$ with an order based on the distance to a certain reference $r$ [1]. Specifically, given a reference $r \in [0,1]$, we define the following total order on the unit interval $[0,1]$:

$$x \preccurlyeq_r y \iff \begin{cases} |x-r| & > & |y-r| \\ & \text{or} & \\ |x-r| & = & |y-r| \text{ and } x \leq y. \end{cases} \tag{10}$$

In words, the inequality $x \preccurlyeq_r y$ holds if $y$ is nearer to the reference $r$ than $x$. Note that $\preccurlyeq_r$ becomes the usual order if $r = 1$. Also, we obtain the usual dual order on $[0,1]$ when $r = 0$. Furthermore, the reference $r$ is always the supremum of $([0,1], \preccurlyeq_r)$ but the infimum depends on the choice of the reference $r$. For example, if the reference is $r = 0.5$ then 0 is the infimum of complete lattice $([0,1], \preccurlyeq_{0.5})$.

Using the distance-based order defined by (10), we propose the distance-based projection autoassociative memories (DBPAMs) for storage and retrieval of vectors on $[0,1]^n$. Formally, let $\mathbb{L} = [0,1]^n = \mathcal{L}_1 \times, \ldots, \times \mathcal{L}_n$ be the complete lattice that inherits the complete lattice structure of $\mathcal{L}_i = [0,1]$ with the distance-based order $\preccurlyeq_{r_i}$ for all $i = 1, \ldots, n$. Given a fundamental memory set $\mathcal{X} = \{\mathbf{x}^1, \ldots, \mathbf{x}^k\} \subset \mathbb{L}$ and references $r_i \in [0,1]$, $i = 1, \ldots, n$, we define the DBPAM $\mathcal{S}_{\mathbf{r}} : [0,1]^n \longrightarrow [0,1]^n$ as follows for all $\mathbf{x} \in [0,1]^n$:

$$\mathcal{S}_{\mathbf{r}}(\mathbf{x}) = \bigcurlywedge_{\xi \in \mathcal{J}_{\mathbf{r}}}^{\mathbf{r}} \mathbf{x}^{\xi}, \quad \text{where} \quad \mathcal{J}_{\mathbf{r}} = \left\{ \xi \in \mathcal{K} : x_i \preccurlyeq_{r_i} x_i^{\xi}, \forall i = 1, \ldots, n \right\}. \tag{11}$$

Note that, if $\mathcal{J}_{\mathbf{r}} = \emptyset$, then $\mathcal{S}_{\mathbf{r}}(\mathbf{x}) = \mathbf{r}$, where $\mathbf{r} = [r_1, \ldots, r_n]^T$ is the vector whose components are the references.

The class of DBPAMs include the PAFMs of Zadeh. Precisely, the max-$C$ PAFM of Zadeh is obtained by considering $\mathbf{r} = [0, 0, \ldots, 0]^T$. Dually, we have the min-$D$ PFAM of Zadeh when $\mathbf{r} = [1, 1, \ldots, 1]^T$. It is not surprisingly that different reference vectors $\mathbf{r} \in [0,1]^n$ yield wildly different DBPAMs models. We would like to point out, however, that the choice of an appropriate reference vector for a given task is an open problem yet. For simplicity, in the following example we consider the constant vector $\mathbf{r} = [0.5, 0.5, \ldots, 0.5]^T$.

**Example 3.** Consider the fundamental memory set $\mathcal{X}$ given by (6) and let $\mathbb{L} = [0,1]^4$ be the complete lattice with the partial order induced by $([0,1], \preccurlyeq_{0.5})$, that is, the order given by (1) and (10) with the reference vector $\mathbf{r} = [0.5, 0.5, 0.5, 0.5]^T$. Given the input vectors $\mathbf{x}$ and $\mathbf{y}$ defined by (7), the DBPAM $\mathcal{S}_{\mathbf{r}}$ produces the outputs

$$\mathcal{S}_{\mathbf{r}}(\mathbf{x}) = \bigcurlywedge_{\xi \in \mathcal{J}_{\mathbf{r}}}^{\mathbf{r}} \mathbf{x}^{\xi} = \mathbf{x}^1 \quad \text{and} \quad \mathcal{S}_{\mathbf{r}}(\mathbf{y}) = \bigcurlywedge_{\xi \in \mathcal{J}_{\mathbf{r}}}^{\mathbf{r}} \mathbf{x}^{\xi} = \mathbf{x}^1, \tag{12}$$

because the inequalities $\mathbf{x} \preccurlyeq_{\mathbf{r}} \mathbf{x}^1$, $\mathbf{x} \not\preccurlyeq_{\mathbf{r}} \mathbf{x}^2$, $\mathbf{x} \not\preccurlyeq_{\mathbf{r}} \mathbf{x}^3$, $\mathbf{y} \preccurlyeq_{\mathbf{r}} \mathbf{x}^1$, $\mathbf{y} \not\preccurlyeq_{\mathbf{r}} \mathbf{x}^2$, and $\mathbf{y} \not\preccurlyeq_{\mathbf{r}} \mathbf{x}^3$ hold true. Note that, in contrast to the CLPAM based on the usual order, the DBPAM

 

$\mathcal{S}_{\mathbf{r}}$ retrieved the fundamental memory $\mathbf{x}^1$ upon presentation of an input $\mathbf{x}$ corrupted by dilative noise as well as an input $\mathbf{y}$ contaminated by erosive noise. Such noise tolerance follows from the inequalities $0 \preccurlyeq_{0.5} 1 \preccurlyeq_{0.5} x$ for any $x \in (0,1)$. Thus, although a vector corrupted by salt and pepper noise is contaminated by both dilative and erosive noise in the ordinary sense (usual order), it corresponds to a vector contaminated only by erosive noise on the complete lattice $([0,1]^n, \preccurlyeq_{0.5})$. This fact suggests that the DBPAM $\mathcal{S}_{\mathbf{r}}$, with the constant reference vector $\mathbf{r} = [0.5, \ldots, 0.5]^T$, can exhibit an excellent performance for the reconstruction of gray-scale images corrupted by salt and pepper noise.

# 4 Concluding Remarks

In this paper, we introduced the class of complete lattice projection autoassociative memories (CLPAMs). Briefly, a CLPAM yields the infimum of the fundamental memories which are greater than or equal to the input. We pointed out that in Theorem 2.1 that a CLPAM is extensive, idempotent, and exhibit optimal absolute storage capacity. Furthermore, we addressed in Theorem 2.2 the noise tolerance of a CLPAM.

From the mathematical point of view, CLPAMs are very well defined on complete lattices. In particular, we proposed the subclass of distance-based projection associative memories (DBPAMs) by considering the unit interval $[0,1]$ equipped with the distance-based ordering $\preccurlyeq_r$ defined by (10). The class of DBPAM includes the projection autoassociative fuzzy memories of Zadeh introduced recently by Santos and Valle [10]. Also, we provided a certain DBPAM which is robust in the presence of salt and pepper noise. In the future, we plan to investigate further the properties the CLPAMs, including the DBPAM models. Furthermore, we intent to apply these novel models of AMs in the reconstruction problem of gray-scale images corrupted by salt and pepper noise.

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