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On the characterization of 3-tessellable graphs

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Abstract. A graph tessellation is a partition of the vertices of the graph into cliques and a graph tessellation cover is a set of graph tessellations that covers the edges of the graph. A graph is 3-tessellable if it has a tessellation cover with three tessellations. The study of graph tessellations is important in quantum computation because the evolution operator of the staggered quantum walk model is obtained from a graph tessellation cover. In this work we establish a characterization on the smallest tessellation cover of a graph G using the chromatic number of its clique graph $\chi(K(G))$ by showing that a diamond-free graph G is 3-tessellable if and only if $\chi(K(G)) \leq 4$ and there is a vertex coloring assignment of $K(G)$ with a special property. As a consequence of such characterization, we obtain a hardness proof for determining if a line graph of a triangle-free graph is 3-tessellable. Moreover, we introduce a special type of edge coloring of a triangle-free graph G which corresponds to a tessellation cover of its line graph. This hardness proof allows us to establish the NP-completeness of this new coloring problem for triangle-free graphs.

Keywords. diamond-free graphs, clique graph, line graph, graph tessellation, staggered quantum walk

1 Introduction

In the last years, the area of quantum walks is steadily increasing possibly because it is considered one of the main tools for building quantum algorithms. Recently, the staggered quantum walk model [7] was proposed with the aim of generalizing previous known models, such as the coined quantum walk model [8]. The dynamics of the staggered model is based on a new concept of graph theory, called graph tessellation cover, which is under development [1].

In this work, we make a new contribution for the recognition of 3-tessellable graphs by characterizing which diamond-free graphs are 3-tessellable. We show that for a diamond-free graph G, G is 3-tessellable if and only if $\chi(G) \leq 4$ and, besides, it is possible to assign a vertex coloring of $K(G)$ with a special property, described ahead (see property P_1). As a

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consequence of such characterization, we obtain a hardness proof for determining if a line graph of a triangle-free graph is 3-tessellable. Next, we present an interesting new way to edge coloring a graph, using stars subgraphs to cover all its P_3's,[4] in such way that if two stars intersect, then their edges do not receive a same color. The previous hardness result allows us to establish the $\mathcal{NP}$-completeness of this new coloring problem for triangle-free graphs.

2 Main definitions

The clique graph $K(G)$ of a graph G has the maximal cliques of G as its vertices and there is an edge between two vertices of $K(G)$ if their corresponding maximal cliques share a neighborhood in G. A similar concept, the line graph $L(G)$ of a graph G has the edges of G as its vertices and there is an edge between two vertices of G if their corresponding edges share a same extreme in G. A coloring (resp. edge-coloring) of a graph G is an assignment of k colors to its vertices (resp. edges) such that two adjacent vertices (resp. two edges which shares an extreme) receive different colors. The smallest value of k for such a graph G admits a coloring (resp. edge-coloring) is denoted by $\chi(G)$ (resp. $\chi'(G)$).

A tessellation of a graph $G = (V, E)$ is a partition of V into vertex-disjoint cliques. A tessellation cover of G with size k is a set of k tessellations of G so that the union of the edges of these vertex disjoint cliques in all these k tessellation covers the edge set $E(G)$. A graph G is k-tessellable if there is a tessellation cover of size k. When k is minimum, we denote it by $T(G)$ (or its tessellation cover number). The t-TESSELLABILITY problem receives a graph G and asks if G is t-tessellable, that is, if $T(G) \leq t$. A maximal clique K of a graph G is said exposed by a tessellation cover $\mathcal{C}$ if $E(K) \nsubseteq E(\mathcal{T})$ for all $\mathcal{T} \in \mathcal{C}$, that is, the edges of K are covered by no tessellation of $\mathcal{C}$. See Ref. [1] for details.

In this paper, for the sake of convenience, we may omit in our proofs and figures the one-vertex cliques in the tessellations. Moreover, we consider only connected graphs with at least three vertices because the tessellation cover of connected components can be considered separately and graphs with one or two vertices can be covered by one tessellations.

The following facts described in [1, 2, 5] are used in this work.

Fact 2.1. If G is a diamond-free graph, then any two maximal clique of G intersect in at most one vertex and $K(G)$ is diamond-free.

Fact 2.2. If G is a triangle-free graph, then $K(L(G)) = G'$, where G' is obtained from G by removing its leaves.

Fact 2.3. Let G be a 3-tessellable diamond-free graph. If C_1 and C_2 are two maximal cliques of G with a common vertex, then C_1 and C_2 cannot be both exposed by a minimum tessellation cover.

Fact 2.4. If G is a 3-tessellable diamond-free graph, then $\chi(K(G)) \leq 4$.

[4] P_3 is the graph •——•——•.

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Note that the last fact states a necessary condition for 3-tessellability of diamond-free graph. In this work, we obtain the sufficient condition and we are able to characterize 3-tessellable diamond-free graphs

3 Tessellation covers on diamond-free graphs

Let us review the results regarding the characterization of 2-tessellable graphs. Portugal [6] showed that a graph is 2-tessellable if and only if its clique graph is 2-colorable. Peterson [5] showed that a graph is the line graph of a bipartite multigraph if and only if its clique graph is 2-colorable. Then, the class of 2-tessellable graphs is the class of line graphs of bipartite multigraphs. This is a full characterization of 2-tessellable graphs. In terms of computational complexity, 2-tessellable graphs are known to be recognizable in linear time [1].

Now we focus on a partial characterization of the class of 3-tessellable graphs, which is an extension of the characterization of 2-tessellable graphs. We define the following properties:

- We say that a 4-colorable graph has **property** $\mathbf{P}_1$ if there is a vertex proper coloring assignment such that at least one color (say c_4) satisfies the following property. For each vertex v with color c_4, if there is a pair of distinct vertices v_1 and v_2 in the neighborhood of v, then the set $\{v, v_1, v_2\}$ does not induce a triangle.

- We say that a 4-colorable graph has **property** $\mathbf{P}_2$ if there is a vertex proper coloring assignment such that at least one color (say c_4) satisfies the following property. For each vertex v with color c_4, if there is a pair of distinct vertices v_1 and v_2 with different colors c_1 and c_2 in the neighborhood of v and the set $\{v, v_1, v_2\}$ induces a triangle, then v is not adjacent to a vertex with the remaining color c_3.

We prove the following result:

Theorem 3.1. *Let G be a diamond-free graph. The following statements are equivalent:*

(i) G is 3-tessellable.

(ii) $K(G)$ is 4-colorable with property P_1.

(iii) $K(G)$ is 4-colorable with property P_2.

Proof. (i) ⇒ (iii) Suppose that G is 3-tessellable and let set $\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3\}$ be a tessellation cover of G. Fact 2.4 implies that $K(G)$ is 4-colorable.

Now we check that $K(G)$ has property P_2. Suppose by contradiction that there is a vertex $v \in K(G)$ with color c_4 that does not have property P_2, that is, there are vertices v_1, v_2 and v_3 with colors c_1, c_2, and c_3, respectively, such that the set $\{v, v_1, v_2\}$ induces a triangle and v_3 is adjacent to v. In G, there exists cliques $C_1 \in \mathcal{T}_1$, $C_2 \in \mathcal{T}_2$, and C corresponding to vertices v_1, v_2, and v; and these cliques are incident to exactly one vertex $w \in G$. Besides, there is another clique $C_3 \in \mathcal{T}_3$ corresponding to vertex v_3 that intersects

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C in one vertex $w_1 \in G$ because v_3 is adjacent to v in $K(G)$. If $w_1 = w$, we have a contradiction because a graph with four maximal cliques incident to a vertex w cannot be 3-tessellable. If $w_1 \neq w$, then the edge w_1w has two maximal cliques incident to vertex w and one maximal clique incident to w_1. This case is also a contradiction because w_1w does not belong to the tessellation cover and G cannot be 3-tessellable.

(iii) $\Rightarrow$ *(ii)* Suppose that $K(G)$ is 4-colorable and has property P_2. Consider a vertex coloring assignment of $K(G)$ with colors c_1, c_2, c_3, and c_4 such that c_4 satisfies the property described in the definition of property P_2, that is, if a vertex $v \in K(G)$ has color c_4 and there are vertices v_1 and v_2 with colors c_1 and c_2 such that $\{v, v_1, v_2\}$ induces a triangle, v cannot be adjacent to a vertex with the remaining color c_3. Then, we can change the color of v from c_4 to the color class v has no neighbors in it. This procedure can be applied to all vertices v with color c_4 that belongs to a triangle. The output is a new coloring assignment that has property P_1.

(ii) $\Rightarrow$ *(i)* Suppose that $K(G)$ is 4-colorable and has property P_1. Consider a vertex coloring assignment of $K(G)$ with colors c_1, c_2, c_3, and c_4 such that c_4 satisfies the property described in the definition of property P_1. We describe a tessellation cover $\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3\}$ of G. Tessellation $\mathcal{T}_1$ contains the maximal cliques of G associated with the vertices in $K(G)$ with color c_1, and the description is analogous for the vertices in $K(G)$ with colors c_2 and c_3.

Now we show that the maximal cliques of G associated with vertices in $K(G)$ of color c_4 can be covered with tessellations $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$. Let C_4 be a maximal clique of G associated with a vertex $v \in K(G)$ of color c_4. Let w_1 and w_2 be two vertices in C_4. There is at most one maximal clique of G incident to each endpoint of the edge w_1w_2 because if there is two maximal cliques C_1 and C_2 both incident to w_1 (or w_2), the vertices v_1, v_2, and v associated with cliques C_1, C_2, and C_4 induces a triangle, violating property P_1.

Since there is at most one maximal clique of G incident to each endpoint of the edge w_1w_2, we can assign integer 1 to w_1w_2 if the incident cliques are associated with vertices of colors c_2 and c_3. It is analogous with integers 2 and 3. We repeat this process and all edges of C_4 receives labels 1, 2, or 3. This procedure can be repeated for all cliques associated with color c_4. Now we extend tessellations $\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3\}$ that were partially described before, and tessellation $\mathcal{T}_1$ covers the edges with label 1, tessellation $\mathcal{T}_2$ covers the edges with label 2, and tessellation $\mathcal{T}_3$ covers the edges with label 3. This is possible because, for each of the remaining non-covered maximal cliques, the vertices of the extremes of the edges with a missing label can be covered by a clique in the corresponding tessellation, that is, in the tessellation with the same label. At this point, all edges of G have been assigned to a tessellation, and the tessellation cover is obtained by addressing any remaining vertex with cliques of size 1. Note that the maximal cliques associated with color c_4 are exposed. □

Corollary 3.1. *Let G be a triangle-free graph. $L(G)$ is 3-tessellable if and only if $\chi(G) \leq 4$.*

Proof. By Fact 2.2, $K(L(G)) = G'$, where G' is obtained from G by removing its leaves. Clearly, $\chi(K(L(G)) = \chi(G') = \chi(G)$ because the missing leaves of G do not modify its chromatic number (disregarding isolated vertices). Note that the line graph $L(G)$ of a triangle-free graph G is diamond-free [3] and, besides, any 4-coloring of G applied to

$K(L(G))$ has property P_1 because $K(L(G))$ is triangle-free. The rest of this proof follows from Theorem 3.1. □

Remark 3.1. Note that the result of Corollary 3.1 also holds for a triangle-free multigraph G. This happens because a multiedge of a graph G corresponds to a true twin vertex in $L(G)$. Besides, we know that the addition of true twins on a t-tessellable graph maintains it t-tessellable [1]. Moreover, neither $\chi(G)$ nor $\chi(K(L(G)))$ is modified by adding multiedges.

Corollary 3.2. 3-TESSELLABILITY *is $\mathcal{NP}$-complete for line graphs of triangle-free graphs.*

Proof. It is known that t-TESSELLABILITY is in $\mathcal{NP}$ [1]. Using Corollary 3.1 we know that the line graph of a triangle-free graph G is 3-tessellable if and only if G has $\chi(G) \leq 4$. Thus, the result follows from the $\mathcal{NP}$-completeness proof of Maffray and Preissmann [4] for deciding if a triangle-free graph G has $\chi(G) \leq 4$. □

Remark 3.2. The only known graph classes t-TESSELLABILITY is $\mathcal{NP}$-complete are: (2,1)-chordal, (1,2), planar, triangle-free, unichord-free, and diamond-free graphs with diameter at most five [1]. Therefore, our results include the line graph of triangle free graphs in such collection of graph classes where t-TESSELLABILITY is hard to solve.

4 Star P_3-cover edge coloring

We introduce a star P_3-cover edge coloring of a graph G. This non-proper edge coloring has star subgraphs in each color class (where all the edges of the stars have the color of that class). We aim to cover all P_3's of the graph using stars subgraph with the restriction that if two stars share an edge, then they need to be in different color classes. Let $\chi'_{SP_3}(G)$ be the minimum number of color classes we need so that G admits a star P_3-cover edge coloring.

Remark 4.1. Consider an extremal case of a star P_3-cover edge coloring of a graph G, where all the stars cover all the vertices of the neighborhood of their centers. If we have a coloring f of G, we may assign the color of each vertex to the stars they center in this extremal case. That is, when two vertices are adjacent, their stars shares an edge, and they need to be in different color classes, what occurs because the two adjacent vertices receives different colors in f. Therefore, $\chi'_{SP_3}(G) \leq \chi(G)$.

Figure 1 (see (a) and (b)) depicts a Mycielski graph, which is a triangle-free graph with $\chi'_{SP_3}(G_1) = 3$ and $\chi(G_1) = 4$ and in (c) and (d) we have the Petersen graph, which has $\chi'_{SP_3}(G_2) = 3$ and $\chi(G_2) = 3$. In (a) we pick the stars with all the neighbors of the centers, the colors of the stars induce a coloring of G on their centers. Therefore, we use $\chi(G) = 4$ colors. However, in (b) we improve this number by not using all the neighbors on the center of the star in vertex k. If we use 3 stars to cover the P_3's which pass by the vertex k instead of only one, but these stars have less number of leaves, it allows us to reduce the number of necessary colors on the assignment. In (c) we can use the 3-coloring of the Petersen graph to establish the star P_3-cover edge coloring with 3-colors. However, in (d) we show that is possible to obtain a different star P_3-cover edge coloring with same number of colors, not using the entire neighborhood in two stars (the ones centered in the vertices a and d). In all cases, we also illustrate the tessellation cover of the line graph associated with the star P_3-cover edge coloring of the graph.

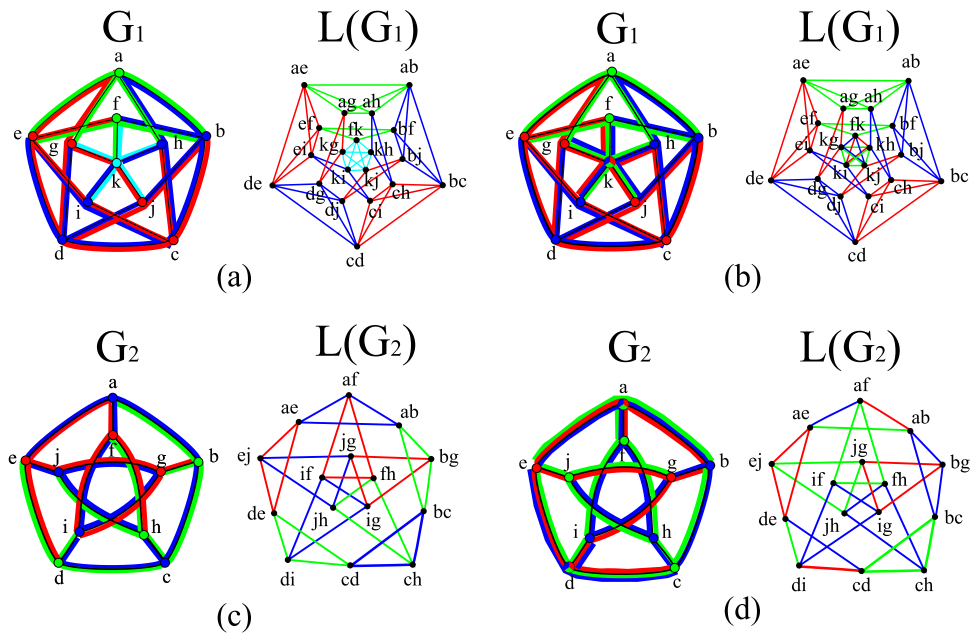

Figure 1: Star P_3-cover edge coloring of a Mycielski graph and Petersen graph.

Theorem 4.1. *A star P_3-cover edge coloring of a triangle-free graph G has a one-to-one correspondence to a tessellation cover of its line graph $L(G)$.*

Proof. Since $L(G)$ is the line graph of a graph G, each P_3 in G corresponds to a distinct edge of $L(G)$. Each star subgraph in any graph G corresponds to a clique in $L(G)$. Moreover, G be triangle-free implies that each clique in $L(G)$ are associated with a star subgraph of G (see Fact 2.2). Note that the triangle-free property is important because if G has a triangle, there is a clique in $L(G)$ that is associated with this triangle, and not to a star subgraph of G.

Let f be a minimum star P_3-cover edge coloring of G. Define $\chi'_{SP_3}(G)$ tessellations of $L(G)$, where each tessellation i contains the cliques of $L(G)$ associated with the edges of the stars in G with the color i. As every P_3 is covered by some stars in G, thus each edge is covered by some tessellation in $L(G)$. Moreover, there is no two cliques in a tessellation sharing a vertex, since it implies that two stars share an edge with the same color, which is a contradiction. Therefore, these $\chi'_{SP_3}(G)$ tessellations form a tessellation cover of G.

Conversely, let g be a minimum tessellation cover of $L(G)$ with $T(G)$ tessellations. Define a star P_3-cover edge coloring f of G as follows. The edges of the stars associated with the clique of a tessellation i of $L(G)$ receive the color i. As all edges of $L(G)$ are covered by a tessellation, thus all P_3's of G are covered by some star. Moreover, if two of such stars share an edge, it means that their cliques share a vertex, that is, they are in different tessellations and the corresponding stars receive different colors. Therefore, g is a star P_3-cover edge coloring of G with $T(G)$ colors. □

Remark 4.2. In the extremal case when all the stars use all the neighbors of their centers, the problem becomes equivalent to coloring of G (see Remark 4.1). Moreover, since it covers all P_3's incident to that vertex, it covers all the edges of the corresponding maximal clique in $L(G)$. Therefore, it is a tessellation cover of $L(G)$ using $\chi(G)$ colors. Since $\chi(G) = \chi(K(G))$ (see Corollary 3.1), the result is consistent because it is known that a tessellation cover that uses a single tessellation for each maximal clique has size $\chi(K(G))$ [1].

Corollary 4.1. *To determine $\chi'_{SP_3}(G)$ of a triangle-free graph G is $\mathcal{NP}$-complete.*

Proof. It follows directly from Corollary 3.1 and Corollary 3.2. □

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