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New approach of Bernoulli and Genocchi Numbers and their associated polynomials, via generalized Fibonacci sequences of order ∞

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Abstract. We are interested here in the application of ∞-generalized Fibonacci sequences (∞-GFS for short), to study some properties of Bernoulli and Genocchi numbers and their related classical numbers and polynomials. That is, properties of this class of sequences, allows us to derive new recursive relations for generating Bernoulli and Genocchi numbers, and their related polynomials.

Keywords. ∞-generalized Fibonacci sequence, Bernoulli numbers, Genocchi numbers.

1 Introduction

The ∞-generalized Fibonacci sequence has been introduced and studied in [4]. Their properties are also studied in [1], [2]. Particularly, their combinatorial properties are exhibited in [6]. The preceding connection between the ∞-generalized Fibonacci sequence and analytic and holomorphic functions, has been considered in [2].

Let $\{v_n\}_{n\in\mathbb{Z}}$ be an ∞-generalized Fibonacci sequence, whose initial values are given by $v_0 = 1$ and $v_{-j} = 0$ for all $j \geq 1$, which is defined by

$$v_{n+1} = \sum_{j=0}^{\infty} a_j v_{n-j} = \sum_{j=0}^{n} a_j v_{n-j}. \tag{1}$$

The associated generating function is given by

$$f(z) = \sum_{n=0}^{\infty} v_n z^n = \frac{1}{Q(z)},$$

where $Q(z)$ is given by

$$Q(z) = 1 - \sum_{j=0}^{\infty} a_j z^{j+1}, \tag{2}$$

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(for more details see [4], [5]). Conversely, let $Q(z)$ be a complex function which is analytic in open disk $D(0;R)$. Suppose that Q takes the following power series form in $D(0;R)$,

$$Q(z) = 1 - \sum_{j=0}^{\infty} a_j z^{j+1}.$$

Since $Q(0) = 1 \neq 0$, $f(z) = 1/Q(z)$ has a Taylor expansion in a certain disk centered at 0, which is of the form

$$f(z) = \sum_{n=0}^{\infty} v_n z^n. \tag{3}$$

The identity $Q(z)f(z) = 1$ implies that we have

$$v_{n+1} = \sum_{j=0}^{\infty} a_j v_{n-j},$$

for all $n \geq 0$, where $v_0 = 1$ and $v_{-j} = 0$ for all $j \geq 1$. Hence, $\{v_n\}_{n\in\mathbb{Z}}$ is an ∞-generalized Fibonacci sequence, whose initial values are given by $v_0 = 1$ and $v_{-j} = 0$ for all $j \geq 1$.

1.1 Truncated ∞-GFS and Bernoulli numbers

The class of Bernoulli numbers $\{B_n\}_{n\geq 0}$ is defined by their associated exponential generating functions as follows

$$\frac{t}{e^t - 1} = \sum_{n=0}^{+\infty} B_n \frac{t^n}{n!}. \tag{4}$$

The left side of (4) can be written as follows

$$\frac{t}{e^t - 1} = \frac{1}{Q(t)}, \text{ where } Q(t) = 1 - \sum_{n=0}^{+\infty} a_n t^{n+1}, \tag{5}$$

with $a_n = -\dfrac{1}{(n+2)!}$. Using expressions (3)-(1), we derive the following property.

Proposition 1.1. *The Bernoulli numbers are expressed in terms of an ∞-GFS as follows*

$$B_n = n! \times v_n, \tag{6}$$

where $\{v_n\}_{n\in\mathbb{Z}}$ is an ∞-GFS, whose coefficients are $a_n = -\dfrac{1}{(n+2)!}$ $(n \geq 0)$ and initial values are $v_0 = 1$ and $v_{-j} = 0$ for all $j \geq 1$.

Expressions (1) and (6), show that the sequence $\{B_n\}_{n\geq 0}$ can be also computed with the aid of the following recursive relation of infinite order

$$B_{n+1} = (n+1)a_0 B_n + n(n+1)a_1 B_{n-1} + \cdots + \frac{(n+1)!}{1!} a_{n-1} B_1 + \frac{(n+1)!}{0!} a_n B_0. \tag{7}$$

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Proposition (1.1) shows that the sequence of Bernoulli numbers $\{B_n\}_{n\geq 0}$ can computed using the linear recursive relation of infinite order (1). In other words, Proposition 1.1 and expression (7), offer a new algorithm for computing recursively the Bernoulli numbers.

On the other hands, starting from the combinatoric aspect of the ∞-GFS, we can also derive a combinatoric expression of the the sequence of Bernoulli numbers. More precisely, Proposition 1.1 and Proposition 3.7 of [2], allow us to obtain the following proposition.

Proposition 1.1. *The Bernoulli numbers are expressed in terms of the combinatoric expression of an ∞-GFS as follows*

$$B_n = n! \times v_n = n! \times \rho(n,0), \tag{8}$$

where $\rho(n,0)$ is the combinatoric expression of the ∞-GFS $\{v_n\}_{n\in\mathbb{Z}}$, whose coefficients are $a_n = -\dfrac{1}{(n+2)!}$ $(n \geq 0)$ and initial values are $v_0 = 1$ and $v_{-j} = 0$ for all $j \geq 1$, given by $\rho(0,0) = 1$ and

$$\rho(n,0) = \sum_{k_0+2k_1+\cdots+nk_{n-1}=n} \frac{(k_0+k_1+\cdots+k_{n-1})!}{k_0!k_1!\cdots k_{n-1}!} a_0^{k_0} a_1^{k_1} \cdots a_{n-1}^{k_{n-1}}, \tag{9}$$

for all $n \geq 1$.

In the best of our knowledge the combinatoric expression (8)-(9) of the Bernoulli numbers are new.

It is well known that there exist a closed connection between Bernoulli numbers and other important classes of classical numbers such as *Genocchi numbers* and *Euler numbers*. Therefore, Propositions 1.1 and 2.1 can also offer a new process for computing these classes of numbers, with the aid of the linear recursive process of order infinite issued from the ∞-GFS properties.

1.2 Truncated ∞-GFS and Genocchi numbers

The class of Genocchi numbers $\{G_n\}_{n\geq 0}$ is defined by the following generating functions,

$$\frac{2t}{e^t+1} = \sum_{n=0}^{+\infty} G_n \frac{t^n}{n!}. \tag{10}$$

The left side of (10) can be written as follows

$$\frac{2t}{e^t+1} = \frac{1}{Q(t)}, \text{ where } Q(t) = 1 - \sum_{n=0}^{+\infty} a_n t^{n+1}. \tag{11}$$

with $a_n = -\dfrac{1}{(n+1)! \times 2}$. Using expressions (3)-(1), we derive the following property.

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Proposition 1.2. *The Genocchi numbers are expressed in terms of an ∞-GFS as follows*

$$G_n = n! \times w_{n-1}, \quad \textit{for } n \geq 1 \tag{12}$$

where $\{w_n\}_{n\in\mathbb{Z}}$ is an ∞-GFS, whose coefficients are $a_n = -\dfrac{1}{(n+1)! \times 2}$ $(n \geq 0)$ and initial values are $w_0 = 1$ and $w_{-j} = 0$ for all $j \geq 1$.

Moreover, linear recursive relation of infinite order (1) and expression (12), show that $w_n = \dfrac{G_{n+1}}{(n+1)!}$ (for $n \geq 1$). Therefore, a simple computation implies that the sequence of Genocchi numbers, satisfies the following recursive relation of infinite order,

$$G_{n+1} = (n+1)na_0G_{n-1} + (n+1)n(n-1)a_1G_{n-2} + \cdots + (n+1)!a_{n-1}G_0, \tag{13}$$

for every $n \geq 0$.

Proposition (1.2) and expression (13) show that the sequence of Genocchi numbers $\{G_n\}_{n\geq 0}$ can be computed with the aid of some recursive relations, issued from the linear recursive relation of infinite order (1). In other words, Proposition 1.2 and expression (13), offer a new algorithm for computing recursively the Genocchi numbers.

Moreover, the combinatoric aspect of the ∞-GFS, can be used for deriving the combinatoric expression of the the sequence of Genocchi numbers. Indeed, Proposition 1.2 and Proposition 3.7 of [2] allow us to obtain the following proposition.

Proposition 1.3. *The Genocchi numbers are expressed in terms of the combinatoric expression of an ∞-GFS as follows*

$$G_n = n! \times w_{n-1} = n! \times \rho(n-1, 0), \quad \textit{for } n \geq 1 \tag{14}$$

where $\rho(n, 0)$ is the combinatoric expression of the ∞-GFS $\{w_n\}_{n\in\mathbb{Z}}$, whose coefficients are $a_n = -\dfrac{1}{(n+1)! \times 2}$ $(n \geq 0)$ and initial values are $w_0 = 1$ and $w_{-j} = 0$ for all $j \geq 1$, given by $\rho(0, 0) = 1$ and (9), for all $n \geq 1$.

It seems for us that the combinatoric expression (14) of the Genocchi numbers is not known in the literature.

If we set $F(t) = \dfrac{2t}{e^t + 1}$, a simple computation shows that,

$$F(-t) = -2t + F(t). \tag{15}$$

Therefore, we have

$$G_0 = 0,\ G_1 = w_0 = 1 \ \ G_{2n+1} = 0 \ \text{ for } \ n \geq 1 \text{ and } G_{2n} = (2n)!w_{2n-1}, \ \text{ for } \ n \geq 1, \tag{16}$$

where $\{v_n\}_{n\geq 0}$ is the ∞-GFS defined by (1). Therefore, expressions (15)-(16) allow us to see that the recursive relation (13), can be reduced to the following

$$G_{2n+2} = \sum_{j=0}^{n-1} a_{2j+1} \frac{(2n+2)!}{[2(n-j)]!} G_{2(n-j)} + (2n+2)!a_{2n}G_1. \tag{17}$$

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There exists a closed connection between Genocchi numbers and other important classes of classical numbers such as *Bernoulli numbers* and *Euler numbers*. Therefore, Propositions 1.2 and 1.3 can also offer a new process for computing these classes of numbers, with the aid of the linear recursive process of order infinite issued from the ∞-GFS properties. Indeed, consider $\{B_n\}_{n\geq 0}$ be the sequence of Bernoulli numbers and $\{E_n\}_{n\geq 0}$ the sequence of Euler numbers. It is well known in the literature that

$$G_{2n} = 2(1-2^{2n})B_{2n} \text{ (Genocchi's Theorem) and } G_{2n} = 2nE_{2n-1},$$

Thus, from expression (14) we derive the following proposition.

Proposition 1.4. *Consider $\{G_n\}_{n\geq 0}$, $\{B_n\}_{n\geq 0}$ and $\{E_n\}_{n\geq 0}$ the sequence of Genocchi, Bernoulli and Euler numbers (respectively). Then, we have*

$$B_{2n} = \frac{(2n)!}{2(1-2^{2n})} w_{2n-1} \text{ and } E_{2n-1} = \frac{(2n)!}{2n} w_{2n-1} \text{ for } n \geq 1. \tag{18}$$

where $\{w_n\}_{n\in\mathbb{Z}}$is the ∞-GFS, whose coefficients are $a_n = -\dfrac{1}{(n+1)!\times 2}$ $(n \geq 0)$ and initial values are $w_0 = 1$ and $w_{-j} = 0$ for all $j \geq 1$.

2 Truncated ∞-GFS and Bernoulli polynomials

In the same way did with Bernoulli and Genocchi numbers, is done to the Bernoulli and Genocchi polynomials and Bernoulli and Genocchi polynomials higher order, but by lack of space we omitted some results.

The sequence of Bernoulli polynomials $\{B_n(x)\}_{n\geq 0}$ can be defined in the literature by various ways. For reason of convenience, we consider the following convenient one based on the notion of generating function,

$$F(t,x) = \frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{+\infty} B_n(x)\frac{t^n}{n!}. \tag{19}$$

From (8)-(19), we show easily that $B_n = B_n(0)$, for every $n \geq 0$.

Since $e^{tx} = \sum_{n=0}^{+\infty} \frac{x^n}{n!} t^n$, expressions (3), (4) and (8)-(9) allow us to derive that

$$F(t,x) = \sum_{n=0}^{+\infty} H_n(x)t^n,$$

where

$$H_n(x) = \sum_{k+p=n} v_k \frac{x^p}{p!} = \sum_{s=0}^{n} \frac{v_{n-s}}{s!} x^s = \sum_{k=0}^{n} \frac{\rho(n-k,0)}{k!} x^k. \tag{20}$$

Therefore, from (19)-(20) we derive the following proposition.

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Proposition 2.1. *The Bernoulli polynomials are expressed in terms of the ∞-GFS as follows*

$$B_n(x) = n! \times H_n(x) = n! \times \sum_{s=0}^{n} \frac{v_{n-s}}{s!} x^s = n! \times \sum_{s=0}^{n} \frac{\rho(n-s,0)}{s!} x^s, \tag{21}$$

where $H_n(x)$ is given by (20), $\rho(n,0)$ (given by (9)) is the combinatoric expression of the ∞-GFS $\{v_n\}_{n\in\mathbb{Z}}$, whose coefficients are $a_n = -\dfrac{1}{(n+2)!}$ $(n \geq 0)$ and initial values are $v_0 = 1$ and $v_{-j} = 0$ for all $j \geq 1$. More precisely, we have the following combinatorial form of the Bernoulli polynomial

$$B_n(x) = \sum_{k=0}^{n} (n-k)!\tbinom{n}{k}\rho(n-k,0)x^k,$$

where the $\rho(n,0)$ are given by (9).

The first few Bernoulli polynomials can be recovered easily as follows. Since

$$a_0 = -\frac{1}{2}\ ,\ a_1 = -\frac{1}{6}\ ,\ a_2 = -\frac{1}{24}\ ,\ a_3 = -\frac{1}{120}\ ;$$

expression (1) implies that $v_0 = 1$; $v_1 = -\frac{1}{2}$; $v_2 = \frac{1}{12}$; $v_3 = 0$ and $v_4 = -\frac{1}{720}$. Thus, a straightforward application of expression (21) gives,

$$B_0(x) = 1\ ,\ B_1(x) = x - \frac{1}{2}\ ,\ B_2(x) = x^2 - x + \frac{1}{6}\ \text{ and } B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.$$

Moreover, we have

$$H_{n+1}(x) = \sum_{k=0}^{n+1} v_{n+1-k}\frac{x^k}{k!} = v_{n+1} + \sum_{s=0}^{n} v_{n-s}\frac{x^{s+1}}{(s+1)!} = v_{n+1} + \int_0^x H_n(t)dt,$$

Therefore, we have

$$B_{n+1}(x) = (n+1)!v_{n+1} + (n+1)\int_0^x \frac{B_n(t)}{n!}dt = B_{n+1} + (n+1)\int_0^x B_n(t)dt.$$

3 Conclusion

In this work we discuss about the application of ∞-generalized Fibonacci sequences (∞-GFS for short) in the to study some properties of Bernoulli and Genocchi numbers and their related classical numbers and polynomials. There are natural questions about this. Can we derive from results of our preceding papers an asymptotic behavior of the Bernoulli numbers? And therefore, derive also an asymptotic behavior of Genocchi and Euler numbers? The computer values show us that the answers is yes. Those are questions for future works.

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