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Structural optimization using the cross-entropy method

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Abstract. This work aims to evaluate the efficiency and robustness of the cross-entropy (CE) method in the context of structural optimization. A two-dimensional truss subject to vertical loads is used as a benchmark test, where one seeks to minimize the structure weight, respecting a structural integrity criterion. The optimal results obtained via CE are compared with reference solutions, obtained via sequential quadratic programming and genetic algorithm. Numerical experiments demonstrate that CE offers a solution for structural optimization that can be very competitive in terms of accuracy and computational efficiency.

Keywords. structural optimization, nonlinear optimization, cross-entropy method

1 Introduction

Structural optimization is an engineering discipline that deals with the minimization of a suitable performance function, seeking to improve the response of a mechanical system of interest. Frequently, the aim is to minimize the structure weight, respecting a suitable criteria of structural integrity. Due to complex geometric configurations and the use of advanced materials, whose behavior is extremely non-linear, this task can be too challenging, requiring the use of very efficient optimization algorithms [2, 7, 8].

This work aims to test the effectiveness and robustness of the cross-entropy (CE) method [5, 6], a relatively new optimization technique, in the context of structural optimization. For this purpose, a structural optimization problem that seeks to minimize the weight of a two-dimensional truss, ensuring its structural integrity, is employed as benchmark.

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The rest of the paper is organized as follows. Section 2 presents the mechanical system of interest and the model equations. In section 3 the reader is introduced to the addressed structural optimization problem and the employed optimization techniques. Numerical results are presented and discussed in section 4, while final considerations are made in section 5.

2 Mathematical modeling

2.1 Structural model

The structural system of interest in this work is two-dimensional truss illustrated in Figure 1, which also show the geometric dimensions of the structure and the employed coordinate system. This truss consists of 11 bars, labeled from 1 to 11, connected through 6 nodes, labeled from 1 to 6, each one with two degrees of freedom, u_e for the horizontal displacement of bar e and v_e for the corresponding vertical displacement, where $e = 1, \cdots, 11$. These bars are made of a single material, with density $\rho = 7900\ kg$ and elastic modulus $E = 210\ GPa$. The kinematic constraints are due to the fixed support on node 1 and the roller support on node 5. Three vertical loads are applied at the nodes 2, 4 and 6, with magnitudes respectively equal to 50 kN, 100 kN and 50 kN. The choice of this structural system is motivated by the fact that it is a standard finite element benchmark, which can be seen in section 4.6 of the reference [3].

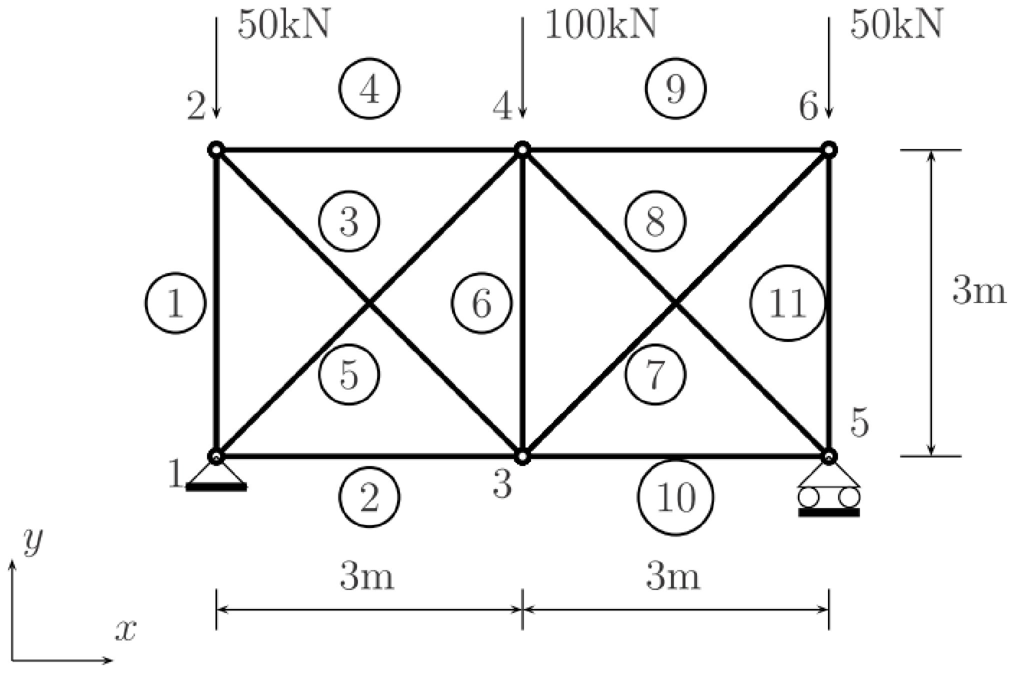

Figure 1: Illustration of the two-dimensional truss (adpated from reference [3]).

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2.2 Equilibrium equations

The equilibrium equations are obtained from the principle of virtual work, being written as the following matrix system

$$\mathbf{K}\,\mathbf{u} = \mathbf{f}\ , \tag{1}$$

where $\mathbf{K}$ is the stiffness matrix, $\mathbf{u}$ is the displacement vector and $\mathbf{f}$ is force vector, which are respectively defined by

$$\mathbf{u} = \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 & u_5 & v_5 & u_6 & v_6 \end{bmatrix}^T\ ,$$

and

$$\mathbf{f} = \begin{bmatrix} 0 & 0 & 0 & -50 & 0 & 0 & 0 & -100 & 0 & 0 & 0 & -50 \end{bmatrix}^T kN\ . \tag{2}$$

Note that due to kinematic constraints, $u_1 = v_1 = v_5 = 0$. Besides that, the stiffness matrix can be written as

$$\mathbf{K} = \sum_{e=1}^{N} \overline{\mathbf{K}}_e\ , \tag{3}$$

where $N = 11$ is the the number of bars and $\overline{\mathbf{K}}_e$ is the elementary stiffness matrix in global coordinates. In local coordinates the elementary stiffness matrix of the bar e is given by

$$\mathbf{K}_e = \frac{A_e E_e}{L_e} \begin{bmatrix} \cos^2\theta_e & \cos\theta_e\sin\theta_e & -\cos^2\theta_e & -\cos\theta_e\sin\theta_e \\ \cos\theta_e\sin\theta_e & \sin^2\theta_e & -\cos\theta_e\sin\theta_e & -\sin^2\theta_e \\ -\cos^2\theta_e & -\cos\theta_e\sin\theta_e & \cos^2\theta_e & \cos\theta_e\sin\theta_e \\ -\cos\theta_e\sin\theta_e & -\sin^2\theta_e & \cos\theta_e\sin\theta_e & \sin^2\theta_e \end{bmatrix}, \tag{4}$$

where A_e is the cross-sectional area, E_e is the material modulus of elasticity, L_e is the element length and θ_e is the angle formed between the bar longitudinal axis and the horizontal axis of the reference system (x axis).

2.3 Structure mass

Each bar of the truss has a circular tubular cross-section with area is given by $A_e = \left(4\,d\,t + t^2\right)\pi/4$, where d is the external diameter and t is the section thickness. In this way, the mass of bar e is given by $m_e(d,t) = \rho\,L_e\,\left(4\,d\,t + t^2\right)\pi/4$ and the total mass of the two-dimensional truss is

$$m(d,t) = \sum_{e=1}^{N} m_e(d,t) = \sum_{e=1}^{N} \rho\,L_e\,\left(4\,d\,t + t^2\right)\pi/4. \tag{5}$$

2.4 Structural integrity criterion

By Hooke's Law the normal stress at the bar e is $\sigma_e = E\,\mathbf{B}_e\,\mathbf{u}_e$, where

$$\mathbf{B}_e = \frac{1}{L_e}\begin{bmatrix} -\cos\theta_e & -\sin\theta_e & \cos\theta_e & \sin\theta_e \end{bmatrix}\ , \tag{6}$$

and the local displacement $\mathbf{u}_e$ is implicitly defined by the local equilibrium equation $\mathbf{K}_e\,\mathbf{u}_e = \mathbf{f}_e$, so that the normal stress can also be written as

$$\sigma_e = E\,\mathbf{B}_e\,\mathbf{K}_e^{-1}\,\mathbf{f}_e. \tag{7}$$

The structural integrity criterion employed here states that (in absolute value) the normal stress at each bar, defined by Eq.(7), is less than or equal to the material yield stress $S_Y = 205\ MPa$. Once σ_e depends on $\mathbf{K}$, that depends on $A_e(d,t)$, the normal stress is a function of d and t and the integrity criterion can be written as

$$|\sigma_e(d,t)| - S_y \leq 0, \qquad e = 1, \cdots, N. \tag{8}$$

3 Structural optimization problem

The structural optimization problem considered here aims to minimize the structure mass, Eq.(5), using d and t as design variables, considering as constraints the inequalities in (8), and a limited set of values for d and t, i.e.,

$$d_{min} \leq d \leq d_{max} \qquad \text{and} \qquad t_{min} \leq t \leq t_{max}. \tag{9}$$

Note that, due to the quadratic terms in the objective function (5), and the inverse dependence of the constraints in (8) with the design variables, this optimization problem is nonlinear.

3.1 Mathematical formulation

The optimization problem seeks to find a vector $\mathbf{x}^\star = (d^\star, t^\star)$ such that

$$\gamma^\star = \max\ \mathcal{S}(\mathbf{x}^\star), \tag{10}$$

where $\gamma^\star$ is the maximum value of the performance function $\mathcal{S}(\mathbf{x})$, that is a penalization of the constrained optimization problem defined by (5), (8) and (9).

3.2 Solution algorithms

Three different optimization techniques are employed in this work to deal with the nonlinear optimization problem in (10), nominally: (i) sequential quadratic programming (SQP), (ii) genetic algorithm (GA), and (iii) cross-entropy (CE) method.

SQP [1] and GA [4] are well-known methods in the structural optimization community. For this reason, and for sake of space limitation, they are not reviewed here.

On the other hand, CE method [5], although being widely used by the combinatorial optimization community, is still little used in continuous optimization problems, such as the one defined by (10). To the best of the authors' knowledge, this is the first time the CE method is used in the context of structural optimization.

The CE method is a heuristic optimization technique based on the minimization of Kullback-Leibler divergence, where the vector $\mathbf{x}$ is randomized following a probability

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Table 1: Normal stress at each bar of the two-dimensional truss.

bar	1	2	3	4	5	6	7	8	9	10	11
σ_e (MPa)	-28.8	16.7	12.1	-6.0	-33.4	-12.1	12.1	-33.4	-6.0	16.7	-28.8

distribution $f(\cdot, \mathbf{v})$ and the optimization problem associated the to the calculation of a rare-event probability. The idea behind this method is to generate a sequence of estimators $(\widehat{\gamma}_l, \widehat{\mathbf{v}}_l)$ such that $\widehat{\gamma}_l \xrightarrow{a.s.} \gamma^\star$ and $f(\mathbf{x}, \widehat{\mathbf{v}}_l) \xrightarrow{a.s.} \delta(\mathbf{x} - \mathbf{x}^\star)$, i.e., the family of distributions $f(\cdot\,, \mathbf{v})$ tends towards a point mass distribution, centered on the global optimum for the optimization problem.

The algorithm can be summarized as follows: (i) Define the number of samples N^s, the number of elite samples N^e, a convergence tolerance `tol`, the maximum of levels l_{max}, a family of probability distributions $f(\cdot, \mathbf{v})$, an initial vector of parameters $\widehat{\mathbf{v}}_0$ for f and set the level counter $l = 0$; (ii) Update level $l = l + 1$; (iii) Generate $\mathbf{X}_1, \cdots, \mathbf{X}_{N^s}$ (iid) samples from $f(\cdot, \widehat{\mathbf{v}}_{l-1})$; (iv) Evaluate performance function $\mathcal{S}(\mathbf{X}_n)$ at samples $\mathbf{X}_1, \cdots, \mathbf{X}_{N^s}$ and sort the results $\mathcal{S}_{(1)} \leq \cdots \leq \mathcal{S}_{(N^s)}$; (v) Update estimators $\widehat{\gamma}_l$ and $\widehat{\mathbf{v}}_l$; (vi) Repeat (ii) — (v) while a stopping criterion is not met. For further details on CE method the reader is encouraged to see [6].

4 Numerical experiments

In the numerical experiments reported below, a penalty factor $p = 10$ is employed and the reference results are computed with SQP[5] and GA[6]. The design variables are limited to the region $50\ mm \leq d \leq 100\ mm$ and $5\ mm \leq t \leq 20\ mm$. The CE algorithm [7] uses a Gaussian distribution truncated over this region.

4.1 Finite element analysis

In order to gain some insight on the truss behavior, a finite element analysis (FEA) is conducted before the structural optimization process. For this, the design variables are assumed as equal to $(d, t) = (100, 20)\ mm$. This FEA reveals that in no bar the structural integrity criterion has been violated, as can be noted in Table 1. The mass total of the truss in this case is $m = 1979\ kg$.

4.2 Optimization experiments

In order to evaluate CE accuracy, a comparison with the reference results obtained via SPQ and GA is done in Table 2. The reader can see that CE performed so well as the other two algorithms, converging with a tolerance of 10^{-6} in only 13 levels (iterations).

[5] `fmincon` routine from MATLAB with default parameters.
[6] `ga` routine from MATLAB with default parameters, and a population with 25 individuals.
[7] CE control parameters are $N^s = 25$, $N^e = 3$, `tol` $= 10^{-6}$ and $l_{max} = 100$.

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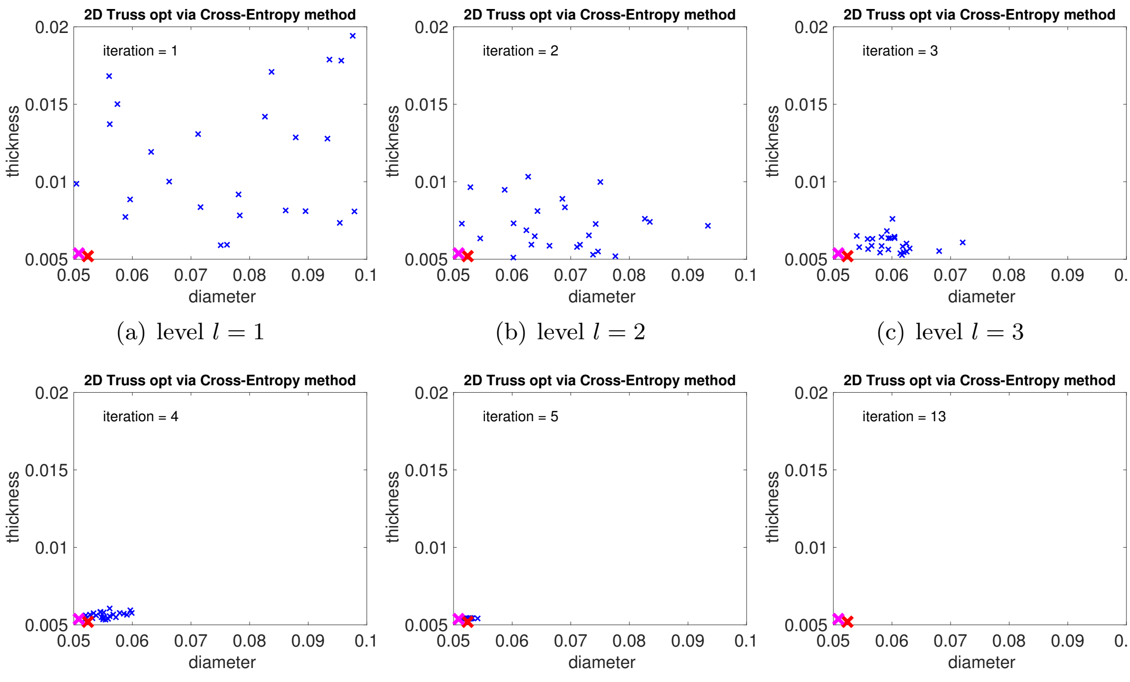
Table 2: Comparison between the results obtained with different optimization techniques.

Method	mass (kg)	$d^\star$ (mm)	$t^\star$ (mm)	CPU time* (sec)
SQP	264	54.4	5.0	0.35
GA	264	50.2	5.4	9.20
CE	265	50.8	5.4	0.48

*MacBook Pro "Core i7" 2.2 GHz 16GB 1333 MHz DDR3

The same table shows that CE method is computationally efficient, once its CPU time is of the same order of magnitude as the SQP, and an order of magnitude smaller than GA. This is very impressive, since SPQ is a gradient-based method, whereas CE has no information about objective function and constraints derivatives. This result shows that CE can be a very competitive heuristic option for structural optimization. An illustration of CE sampling of the domain, at different levels (iterations), is presented in Figure 2, where the red cross is the SQP reference solution, and the magenta cross corresponds to CE solution.



(a) level $l = 1$ (b) level $l = 2$ (c) level $l = 3$

(d) level $l = 4$ (e) level $l = 5$ (f) level $l = 13$

Figure 2: Illustration of CE sampling of the domain at different levels (iterations). The red cross is the SPQ reference solution, and the magenta cross corresponds to CE solution.

5 Final remarks

This work addresses a numerical study evaluating the effectiveness and robustness of the cross-entropy method in the context of structural optimization. The results show that the evaluated method is very competitive, having a much better performance than genetic algorithm in terms of processing time. In comparison to the sequential quadratic programming, cross-entropy has comparable processing time, proving to be a very appealing tool for optimization problems, especially when the use of gradients is impractical.

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