Proceeding Series of the Brazilian Society of Computational and Applied Mathematics, v. 7, n. 1, 2020.

Trabalho apresentado no XXXIX CNMAC, Uberlândia - MG, 2019.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

A Petrov-Galerkin Multiscale Hybrid-Mixed method for the Darcy Equation on Polytopes

Honório Fernando [1]
Universidade Federal Fluminense - UFF
Larissa Martins [2]
Laboratório Nacional de Computação Científica - LNCC
Frédéric Valentin [3]
Laboratório Nacional de Computação Científica - LNCC

1 Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be an open, bounded, polygonal domain with a Lipschitz boundary $\partial\Omega$. We consider the second order elliptic problem defined by finding $u \in H_0^1(\Omega)$ such that

$$(A\nabla u, \nabla v)_\Omega = (f, v)_\Omega \quad \text{for all } v \in H_0^1(\Omega)\,, \tag{1}$$

where $f \in L^2(\Omega)$ and $A \in L^\infty(\Omega)^{d\times d}$ is a symmetric, uniformly elliptic tensor in Ω and may involve multiscale features. In this work, we modify the Multiscale Hybrid-Mixed (MHM) method [1] to propose a new multiscale finite element method to approximate (1). Its construction starts from a Petrov-Galerkin formulation for the Lagrange multiplier variable searched in a polynomial space enriched with residual functions. As a result, jumping terms are added to the original MHM method. These extra terms preserve the accuracy and the overall properties of the original MHM method, while yield underlying symmetric positive definite linear systems. Some notations are needed at that point. We start by introducing $\mathscr{P}$, a collection of closed, bounded, disjoint polytopes, denoted K, such that $\bar{\Omega} = \cup_{K\in\mathscr{P}} K$. The shape of the polytopes K is, a priori, arbitrary, but we will suppose they satisfy a minimal angle condition (see [2]). For each $K \in \mathscr{P}$, $\boldsymbol{n}^K$ denotes the unit outward normal to ∂K. We also introduce $\partial\mathscr{P}$, the set of boundaries ∂K, with $K \in \mathscr{P}$, and $\mathcal{E}$ the following set of faces of $\mathscr{P}$,

$$\mathcal{E} := \{E = K \cap K' \text{ or } K \cap \partial\Omega \,:\, K, K' \in \mathscr{P}, \text{ and it is not reduced to a } d-1 \text{ variety}\}\,.$$

Also, let $(\cdot,\cdot)_D$ be the $L^2(D)$-inner product and $\langle\cdot,\cdot\rangle_{\partial D}$ the duality-product between $H^{-\frac{1}{2}}(\partial K)$ and $H^{\frac{1}{2}}(\partial K)$. The following spaces will be useful in the sequel; $V := H^1(\mathscr{P})\,, V_0 :=$

[1]e-mail: honorio.jf@gmail.com
[2]This author is funded by CAPES. e-mail: larissam@lncc.br
[3]This author is funded by CNPq and CAPES. email: valentin@lncc.br

$\{v \in L^2(\Omega) : v|_K \in \mathbb{P}_0(K),\ \forall K \in \mathscr{P}\}$, $\Lambda := \{\boldsymbol{\tau} \cdot \boldsymbol{n}^K|_{\partial K} : \boldsymbol{\tau} \in H(div, \Omega),\ \forall K \in \mathscr{P}\}$ and $\tilde{V} = V \cap L^2_0(\mathscr{P})$. The finite dimensional space $\Lambda_H \subset \Lambda$ stands for the space of discontinuous polynomial functions on (sub)faces $F \subset E$ of degree $l \geq 0$, and $\tilde{V}_h \subset \tilde{V}$ the space of piecewise continuous polynomial functions in each K of degree $k \geq l + d$. Here $\mathbb{P}_0(K)$ is the piecewise constant function space in each $K \in \mathscr{P}$, and $L^2_0(\mathscr{P})$ the space of functions with zero mean value in $K \in \mathscr{P}$. The usual jumping operator is denoted by $[\![\cdot]\!]$.

2 The Petrov-Galerkin MHM method

Let T_h and $\hat{T}_h$ be the following local bounded linear mappings:

- $\forall \mu \in \Lambda$, $\exists!\ T_h\mu \in \tilde{V}_h$ s.t. $(A\nabla T_h\mu, \nabla v_h)_K = (\mu, v_h)_{\partial K},\ \forall v_h \in \tilde{V}_h$ and $K \in \mathscr{P}$;
- $\forall q \in L^2(\Omega)$, $\exists!\ \hat{T}_h q \in \tilde{V}_h$ s.t. $(A\nabla \hat{T}_h q, \nabla v_h)_K = (q, v_h)_K, \forall v_h \in \tilde{V}_h$ and $K \in \mathscr{P}$,

and $a_h(\cdot,\cdot)$, $b_h(\cdot,\cdot)$, $c_h(\cdot,\cdot)$ and $f_h(\cdot,\cdot)$, $\hat{f}_h(\cdot,\cdot)$ are the following bilinear and linear forms, respectively,

$$\begin{cases} a_h : \Lambda \times \Lambda \to \mathbb{R} \quad \text{where} \quad a_h(\lambda, \mu) = \displaystyle\sum_{K \in \mathscr{P}} \langle \mu, T_h \lambda \rangle_{\partial K} - \sum_{E \in \mathcal{E}_H} \tau_E\,([\![T_h\,\mu]\!], [\![T_h\,\lambda]\!])_E\,, \\ b_h : \Lambda \times V_0 \to \mathbb{R} \quad \text{where} \quad b_h(\lambda, v_0) = \displaystyle\sum_{K \in \mathscr{P}} \langle \lambda, v_0 \rangle_{\partial K} - \sum_{E \in \mathcal{E}_H} \tau_E\,([\![T_h\,\lambda]\!], [\![v_0]\!])_E\,, \\ c_h : V_0 \times V_0 \to \mathbb{R} \quad \text{where} \quad c_h(u_0, v_0) = -\displaystyle\sum_{E \in \mathcal{E}_H} \tau_E\,([\![u_0]\!], [\![v_0]\!])_E\,, \end{cases}$$

$$\begin{cases} f_h : V_0 \to \mathbb{R} \quad \text{where} \quad f_h(v_0) = -\displaystyle\sum_{K \in \mathscr{P}} (f, v_0)_K + \sum_{E \in \mathcal{E}_H} \tau_E\,([\![\hat{T}_h\,f]\!], [\![v_0]\!])_E\,, \\ \hat{f}_h : \Lambda \to \mathbb{R} \quad \text{where} \quad \hat{f}_h(\mu) = -\displaystyle\sum_{K \in \mathscr{P}} \langle \mu, \hat{T}_h f \rangle_{\partial K} + \sum_{E \in \mathcal{E}_H} \tau_E\,([\![T_h\,\mu]\!], [\![\hat{T}_h\,f]\!])_E\,. \end{cases}$$

Here $\tau_E := \frac{\alpha\,A_{min}}{2\,\mathcal{H}_E}$, $\mathcal{H}_E$ is the diameter of $E \in \mathcal{E}$, α is a positive constant independent of the mesh parameters and A_{min} is the minimum value of the spectrum of A. The new method is given as follows: Find $(\lambda_H, u_0^h) \in \Lambda_H \times V_0$ such that

$$\begin{cases} a_h(\lambda_H, \mu_H) + b_h(\mu_H, u_0^h) = \hat{f}_h(\mu_H) \quad \text{for all } \mu_H \in \Lambda_H\,, \\ b_h(\lambda_H, v_0) + c_h(u_0^h, v_0) = f_h(v_0) \quad \text{for all } v_0 \in V_0\,. \end{cases}$$

References

[1] R. Araya, C. Harder, D. Paredes, and F. Valentin. Multiscale Hybrid-Mixed Method. *SIAM J. Numer. Anal.*, 51(6):3505-3531, 2013.

[2] G. R. Barrenechea, F. Jaillet, D. Paredes and Valentin, F. The Multiscale Hybrid Mixed Method in General Polygonal Meshes *Report HAL-02054681, 2019*

