Proceeding Series of the Brazilian Society of Computational and Applied Mathematics, v. 7, n. 1, 2020.

Trabalho apresentado no XXXIX CNMAC, Uberlândia - MG, 2019.

Proceeding Series of the Brazilian Society of Computational and Applied Mathematics

Time-space fractional for the Stefan model

Eliana Contharteze Grigoletto [1]
Departamento de Bioprocessos e Biotecnologia, FCA, UNESP, Botucatu, SP
Rubens de Figueiredo Camargo [2]
Departamento de Matemática, FC, UNESP, Bauru, SP
Michael Vynnycky [3]
KTH Royal Institute of Technology, Stockholm, Sweden

Abstract. In this paper, we solve a particular time-space fractional Stefan problem including fractional order derivatives in time and space variables in the Fourier heat conduction equation. For this, we consider fractional time derivative of order $\alpha \in (0, 1]$ and fractional space derivative of order 2β with $\beta \in \left(\frac{1}{2}, 1\right]$, both in the Caputo sense. Including time and space fractional derivatives, the melt front advances as $s \sim t^{\xi}$, where $\xi = \xi(\alpha, \beta)$, and we can recover sub-diffusion, classical diffusion and super-diffusion behaviors. The result for the proposed problem depends on the choice of order of fractional derivatives α and β provided that the choice satisfies the relation $\alpha = \frac{2\beta}{1+\beta}$.

Keywords. Stefan problems, Time-space fractional, Caputo derivative

1 Introduction

In natural and industrial processes, a certain material undergoes phase change from one phase to another, it melts or solidifies and the solid-liquid interface or phase-change is called as the Stefan problem. The Stefan problem [3,12,13] is well studied in the literature and presents various and important applications in many physical processes.

Mathematically describing solidification or melting is a moving boundary problem [3]. Particularly, in the solution of the one-dimensional phase-change melting problem, the melt front advances as $s(t) = k_{\xi} t^{\xi}$, with a time exponent $0 < \xi < \frac{1}{2}$, $\xi = \frac{1}{2}$ or $\xi > \frac{1}{2}$, the so-called sub-diffusion, classical diffusion or super-diffusion, respectively. The phenomenon of anomalous diffusion (when $\xi \neq \frac{1}{2}$) is observed in various complex systems, including polymers, biopolymers, proteins, porous media, macromolecules transport in biological cells, turbulent flow, among others [7, 8]. Arbitrary order calculus, known as fractional calculus, have obtained important results to many models related to the complex systems modeling [6].

[1]eliana.contharteze@unesp.br.
[2]rubens.camargo@unesp.br.
[3]michaelv@kth.se.

 © 2020 SBMAC

Stefan problems with fractional derivatives have been dealt with by some authors in recent years. Fractional order derivatives in time and space variables were studied by Voller [14], but the study considered the fractional order derivatives in time and space variables separately. The time-fractional diffusion equation, in the Caputo sense, was used by Roscani [11] in fractional free-boundary Stefan problem investigation.

The paper is organized as follows. A brief review of fractional calculus is treated in Section 2. Section 3 is devoted to the explicit solution to a particular Stefan problem with the fractional approach.

2 Preliminary concepts of fractional calculus

The Caputo left-sided fractional derivative of $f \in \mathrm{C}^n[0,b]$ with respect to y, of order $\alpha > 0$ with $\alpha \notin \mathbb{N}$, with starting point 0^+ is given by

$$\left({}^{\mathrm{C}}_{y}\mathrm{D}^{\alpha}_{0+}f\right)(y) := \frac{1}{\Gamma(n-\alpha)}\left(\int_a^y \frac{f^{(n)}(\tau)}{(y-\tau)^{1-n+\alpha}}\,\mathrm{d}\tau\right), \qquad \text{for } 0 \leq y \leq b, \tag{1}$$

where[4] $n = [\alpha] + 1$ and $\Gamma(\cdot)$ denotes the gamma function.

If $\alpha = n \in \mathbb{N}$, then $\left({}^{\mathrm{C}}_{y}\mathrm{D}^{n}_{0+}f\right)(y) := f^{(n)}(y)$.

Property 2.1. Let $\Omega = [0,b]$ be an interval on the real axis $\mathbb{R}$, $\mu > -1$ and $\alpha > 0$, with $\alpha \notin \mathbb{N}$, then

$${}^{\mathrm{C}}_{x}\mathrm{D}^{\alpha}_{0+}x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}x^{\mu-\alpha}, \qquad \text{for } \mu > [\alpha] \text{ and } 0 \leq x \leq b, \tag{2}$$

and

$${}^{\mathrm{C}}_{x}\mathrm{D}^{\alpha}_{0+}x^{\mu} = 0, \qquad \text{for } \mu = 0,1,2,...,[\alpha] \text{ and } 0 \leq x \leq b. \tag{3}$$

The Laplace transform theory is a sophisticated way to solve a class of differential equations because leads a starting problem to an auxiliary problem. In this way our starting problem is converted into another one of seemingly simpler solution [1].

Let $f(t)$ be a real function of time variable $t \geq 0$. The Laplace transform of f, denoted by $\mathscr{L}[f](s) = \mathrm{F}(s)$, is defined by

$$\mathscr{L}[f](s) = \mathrm{F}(s) = \int_0^\infty e^{-st}f(t)\mathrm{d}t, \tag{4}$$

whenever the integral converges for[5] $\mathcal{R}[s] \geq \sigma > 0$, where $s = \sigma + i\tau$, with σ and τ real numbers, and $\mathrm{F}(s) = 0$ for $\sigma < 0$.

The inverse Laplace transform of $\mathrm{F}(s)$ is given by the formula

$$\mathscr{L}^{-1}[\mathrm{F}](t) = f(t) = \frac{1}{2\pi i}\lim_{\tau\to\infty}\int_{\sigma-i\tau}^{\sigma+i\tau} e^{st}\,\mathrm{F}(s)\mathrm{d}s, \tag{5}$$

[4] $[\mu]$ indicates the integer part of μ.
[5] $\mathcal{R}[s]$ indicates the real part of s.

DOI: 10.5540/03.2020.007.01.0445 © 2020 SBMAC

where σ is large enough that $\mathrm{F}(s)$ is defined for $\mathcal{R}[s] \geq \sigma > 0$.

The Laplace transform of the Caputo fractional derivative of $\phi(x,t)$ with respect to t is given by

$$\mathscr{L}\left[{}_{t}^{\mathrm{C}}\mathrm{D}_{0+}^{\alpha}\phi\right](x,s) = s^{\alpha}\mathscr{L}[\phi](x,s) - \sum_{k=0}^{[\alpha]} s^{\alpha-1-k}\frac{\partial^{(k)}\phi}{\partial t^{k}}\left(x,0^{+}\right). \tag{6}$$

The classical Mittag-Leffler function $\mathrm{E}_{\alpha}(x)$ was introduced by Mittag–Leffler [10] and defined by

$$\mathrm{E}_{\alpha}(x) = \sum_{j=0}^{\infty}\frac{x^{j}}{\Gamma(\alpha j+1)}, \qquad \text{for } x \in \mathbb{R} \text{ and } \alpha > 0. \tag{7}$$

The Mittag-Leffler function $\mathrm{E}_{\alpha,\beta}(x)$, generalizing the one in equation (7), is defined by

$$\mathrm{E}_{\alpha,\beta}(x) = \sum_{j=0}^{\infty}\frac{x^{j}}{\Gamma(\alpha j+\beta)}, \qquad \text{for } x,\ \beta \in \mathbb{R} \text{ and } \alpha > 0. \tag{8}$$

The following more simple special relations involving the Mittag-Leffler functions $\mathrm{E}_{2\alpha}(x)$ and $\mathrm{E}_{2,2}(x)$ are valid and were proved by Camargo [2], p. 73 and Grigoletto [4] p. 45, respectively:

$$\mathrm{E}_{2\alpha}(x) = \frac{1}{2}\left[\mathrm{E}_{\alpha}(x^{\frac{1}{2}}) + \mathrm{E}_{\alpha}(-(x^{\frac{1}{2}}))\right] \Rightarrow \mathrm{E}_{2}(x) = \frac{e^{\sqrt{x}}+e^{-\sqrt{x}}}{2}. \tag{9}$$

$$\mathrm{E}_{2,2}(x) = \frac{e^{\sqrt{x}}-e^{-\sqrt{x}}}{2\sqrt{x}}. \tag{10}$$

Theorem 2.1. Let $\Lambda = [0,b]$ be an interval on the real axis $\mathbb{R}$, n a natural number and $\alpha \in \mathbb{R}$, $\left(\frac{n-1}{n} < \alpha \leq 1\right)$. Then the general solution of the linear sequential differential equation

$$\left({}_{x}^{\mathrm{C}}\mathrm{D}_{a+}^{n\alpha}\varphi\right)(x) - \lambda\varphi(x) = 0 \tag{11}$$

can be written as

$$\varphi_{\alpha}(x) = \sum_{k=1}^{n} c_{k}x^{k-1}\mathrm{E}_{n\alpha,k}\left(\lambda x^{n\alpha}\right), \tag{12}$$

where $\{c_k\}_{k=1}^{n}$ are arbitrary constants and $x \in \Lambda$.

Proof. See Grigoletto et al. [5], Theorem 2 and Corollary 2.

3 Fractional one-phase Stefan problem

To solve mathematical model of classical melting Stefan problem in one-phase and one-dimensional semi-infinite domain $x \geq 0$, as described by Mitchell and Vynnycky [9], with fractional calculus, the Fourier heat conduction is replaced by

$${}_{t}^{\mathrm{C}}\mathrm{D}_{0+}^{\alpha}\,\mathrm{T}(x,t) = {}_{x}^{\mathrm{C}}\mathrm{D}_{0+}^{2\beta}\,\mathrm{T}(x,t), \ \ 0 < x < s(t), \ \ t > 0, \tag{13}$$

 © 2020 SBMAC

where $\alpha \in (0,1]$, $\beta \in \left(\frac{1}{2},1\right]$, $\mathrm{T}(x,t)$ is the dimensionless temperature on the surface x at time t, and as time advances, $x = s(t)$ is the solid-liquid interface. Equation (13) is subject to the initial conditions

$$s(0) = 0, \qquad \mathrm{T}(x,0) = 0, \tag{14}$$

boundary conditions

$$\mathrm{T}(0,t) = 1, \qquad \mathrm{T}(s(t),t) = 0, \tag{15}$$

and Stefan condition

$$\sigma \frac{\mathrm{d}s}{\mathrm{d}t} = -{}_x^{\mathrm{C}}\mathrm{D}_{0+}^{\beta}\,\mathrm{T}(x,t)\Big|_{x=s(t)}, \text{ with } \sigma > 0, \tag{16}$$

where σ is the reciprocal of Stefan number (St).

Here we solve equation (13) subject to the conditions in (14)–(16) by means of an integral transform method [1]. Applying the Laplace transform to (13) and boundary condition $\mathrm{T}(0,t) = 1$ with respect to t, by means of (6) and (12), we reduce the problem in equation (13) to the following expression

$$\mathrm{T}(x,s) = \frac{1}{s}\,\mathrm{E}_{2\beta}\left(s^{\alpha}x^{2\beta}\right) + c_2\,x\,\mathrm{E}_{2\beta,2}\left(s^{\alpha}x^{2\beta}\right). \tag{17}$$

Substituting $\alpha = \beta = 1$ into equation (17) and using relations (9) and (10), we have

$$\mathrm{T}(x,s) = \frac{1}{s}\left(\frac{e^{\sqrt{s}x} + e^{-\sqrt{s}x}}{2}\right) + c_2\left(\frac{e^{\sqrt{s}x} - e^{-\sqrt{s}x}}{2\sqrt{s}}\right),$$

which corresponds to the expression for Laplace transform of Fourier heat conduction equation with derivatives of integer order. One can argue that c_2 must be $-\frac{1}{\sqrt{s}}$, as otherwise, the boundary condition $\mathrm{T}(s(t),t) = 0$ cannot be satisfied, then, in accordance with this, we rewrite equation (17) as follows

$$\mathrm{T}(x,s) = \frac{1}{s}\,\mathrm{E}_{2\beta}\left(s^{\alpha}x^{2\beta}\right) - s^{\frac{\alpha}{2\beta}-1}\,x\,\mathrm{E}_{2\beta,2}\left(s^{\alpha}x^{2\beta}\right). \tag{18}$$

Specifically, we take $c_2 = -s^{\frac{\alpha}{2\beta}-1}$ due to the derivation of the boundary $s(t)$ which will be used later to solve the problem. Substituting the Mittag-Leffler function defined by the series in equations (7) and (8) into equation (18) and taking the inverse Laplace transform we arrive at

$$\mathrm{T}(x,t) = 1 - \left[\sum_{j=0}^{\infty}\frac{\left(\frac{x}{t^{\frac{\alpha}{2\beta}}}\right)\left(\frac{x^{2\beta}}{t^{\alpha}}\right)^{j}}{\Gamma(2\beta j+2)\,\Gamma\left(-\alpha j - \frac{\alpha}{2\beta}+1\right)} - \sum_{j=1}^{\infty}\frac{\left(\frac{x^{2\beta}}{t^{\alpha}}\right)^{j}}{\Gamma(2\beta j+1)\,\Gamma(1-\alpha j)}\right]. \tag{19}$$

The moving boundary $s(t)$ at any time t must be proportional to $t^{\frac{\alpha}{2\beta}}$, that is,

$$s(t) = \gamma\,t^{\frac{\alpha}{2\beta}}, \tag{20}$$

 © 2020 SBMAC

where γ is a constant to be determined.

Next we will find the solution to the equation (13) with the conditions (14)–(16). Our solution must satisfy the boundary condition $\mathrm{T}(s(t),t) = 0$ in equation (15). So if we substitute $s(t)$ in equation (20) into equation (19) and introduce a constant η on the function $\mathrm{T}(x,t)$ in equation (20), we find explicit solution of the problem, given by

$$\mathrm{T}(x,t) = 1 - \frac{1}{\eta}\left[\sum_{j=0}^{\infty} \frac{\left(\frac{x}{t^{\frac{\alpha}{2\beta}}}\right)\left(\frac{x^{2\beta}}{t^{\alpha}}\right)^{j}}{\Gamma(2\beta j+2)\,\Gamma\left(-\alpha j - \frac{\alpha}{2\beta}+1\right)} - \sum_{j=1}^{\infty} \frac{\left(\frac{x^{2\beta}}{t^{\alpha}}\right)^{j}}{\Gamma(2\beta j+1)\,\Gamma(1-\alpha j)}\right], \quad (21)$$

where

$$\eta = \sum_{j=0}^{\infty} \frac{\gamma^{2\beta j+1}}{\Gamma(2\beta j+2)\,\Gamma\left(-\alpha j - \frac{\alpha}{2\beta}+1\right)} - \sum_{j=1}^{\infty} \frac{\gamma^{2\beta j}}{\Gamma(2\beta j+1)\,\Gamma(1-\alpha j)}. \quad (22)$$

If we now consider the following relation between the fractional order α and β given by

$$\alpha = \frac{2\beta}{1+\beta}, \quad (23)$$

then the expression of coefficient γ can be derived from equation (16), where we use the Property 2.1. The expression satisfies the transcendental equation

$$\sigma\gamma^{\beta+1}\frac{\alpha}{2\beta}\,\eta = \left[\sum_{j=0}^{\infty} \frac{\gamma^{2\beta j+1}}{\Gamma(2\beta j+2-\beta)\,\Gamma\left(-\alpha j - \frac{\alpha}{2\beta}+1\right)} - \sum_{j=1}^{\infty} \frac{\gamma^{2\beta j}}{\Gamma(2\beta j+1-\beta)\,\Gamma(1-\alpha j)}\right]. \quad (24)$$

Particularly, when $\alpha \to 1$ and $\beta \to 1$, the solution in equation (21) coincides with the solution presented in [9], given by

$$\mathrm{T}(x,t) = 1 - \frac{\mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right)}{\mathrm{erf}(\mu)}, \quad (25)$$

where $\mathrm{erf}(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-\tau^2}\,\mathrm{d}\tau$ is error function, $s(t) = 2\mu\sqrt{t}$, where μ satisfies the transcendental equation

$$\sqrt{\pi}\sigma\mu\,\mathrm{erf}(\mu)\,e^{\mu^2} = 1. \quad (26)$$

4 Conclusions

In this paper, we presented the explicit solution, by means of an integral transform method and by the aid of calculations, to a fractional approach to a particular mathematical model of the classical melting Stefan problem in one-phase and one-dimensional semi-infinite domain, where we considered fractional order derivatives in time and space variables in the Caputo sense. We have recovered the results of integer derivative model proposed by Mitchell and Vynnycky [9] as a special case and we observe that for choices

DOI: 10.5540/03.2020.007.01.0445 © 2020 SBMAC

of α and β satisfying the relation in equation (23), if $\alpha < \beta$, $\alpha = \beta$ or $\alpha > \beta$, sub-diffusion, classical diffusion or super-diffusion behaviors are obtained, respectively. The future work would be to use the fractional space derivative in the Riesz sense (space-fractional Laplacian) and the fractional time derivative in the Caputo sense in order to solve Stefan problems in one-dimensional unbounded space.

Acknowledgment

The authors thank the research group CF@FC for important and productive discussions.

References

[1] R. E. Bellman and R. S. Roth. *The Laplace Transform.* World Scientific, Singapore, 1984.

[2] R. F. Camargo. Fractional Calculus and Applications. Tese de Doutorado, Imecc, Unicamp, Campinas, 2009.

[3] J. Crank. *Free and Moving Boundary Problems.* Clarendon Press, Oxford, 1984.

[4] E. C. Grigoletto. Fractional Differential Equations and the Mittag-Leffler Functions. Tese de Doutorado, Imecc, Unicamp, Campinas, 2014.

[5] E. C. Grigoletto, E. C. de Oliveira, and R. F. Camargo. Linear fractional differential equations and eigenfunctions of fractional differential operators, *Comp. Appl. Math.*, 37:1012–1026, 2018.

[6] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. *Theory and Applications of Fractional Differential Equations.* Elsevier, Amsterdam, 2006.

[7] G. Margolin, and B. Berkowitz. Spatial behavior of anomalous transport, *Phys. Rev., E*, 65:1–11(031101), 2002.

[8] R. Metzler, and J. Klafter. The random walk's guide to anomalous diffusion: a fractional dynamics approach, *Phys. Rep.*, 339:1–77, 2000.

[9] S. L. Mitchell, and M. Vynnycky. Finite-difference methods with increased accuracy and correct initialization for one-dimensional Stefan problems, *Appl. Math. Comput.*, 215:1609–1621, 2009.

[10] G. M. Mittag-Leffler. Sur l'intégrale de Laplace-Abel, *C. R. Acad. Sci. Paris, Sér. II*, 136:937–939, 1902.

[11] S. D. Roscani. Hopf lemma for the fractional diffusion operator and its application to a fractional free-boundary problem, *J. Math. Anal. Appl.*, 434:125–135, 2016.

 © 2020 SBMAC

[12] B. Šarler. Stefan's work on solid-liquid phase change, *Eng. Anal. Bound. Elem.*, 16:83–92, 1995.

[13] J. Stefan. Über die Theorie der Eisbildung, *Monatsh Math. und Physik*, 1:1–6, 1890.

[14] V. R. Voller. Fractional Stefan problems, *Int. J. Heat Mass Transfer*, 74:269–277, 2014.

DOI: 10.5540/03.2020.007.01.0445 © 2020 SBMAC